Introduction to Financial Econometrics
Chapter 5 Introduction to Portfolio Theory

Eric Zivot
Department of Economics
University of Washington

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1 Introduction to Portfolio Theory

Consider the following investment problem. We can invest in two non-dividend paying stocks A and B over the next month. Let R_A denote monthly return on stock A and R_B denote the monthly return on stock B. These returns are to be treated as random variables since the returns will not be realized until the end of the month. We assume that the returns R_A and R_B are jointly normally distributed and that we have the following information about the means, variances and covariances of the probability distribution of the two returns:

$$
\begin{aligned}
\mu_A &= E[R_A],\ \sigma_A^2 = Var(R_A), \\
\mu_B &= E[R_B],\ \sigma_B^2 = Var(R_B), \\
\sigma_{AB} &= Cov(R_A, R_B).
\end{aligned}
$$

We assume that these values are taken as given. We might wonder where such values come from. One possibility is that they are estimated from historical return data for the two stocks. Another possibility is that they are subjective guesses.

The expected returns, μ_A and μ_B, are our best guesses for the monthly returns on each of the stocks. However, since the investments are random we must recognize that the realized returns may be different from our expectations. The variances, σ_A^2 and σ_B^2, provide measures of the uncertainty associated with these monthly returns. We can also think of the variances as measuring the risk associated with the investments. Assets that have returns with high variability (or volatility) are often thought to be risky and assets with low return volatility are often thought to be safe. The covariance σ_{AB} gives us information about the *direction* of any linear dependence between returns. If $\sigma_{AB} > 0$ then the returns on assets A and B tend to move in the same direction; if $\sigma_{AB} < 0$ the returns tend to move in opposite directions; if $\sigma_{AB} = 0$ then the returns tend to move independently. The strength of the dependence between the returns is measured by the correlation coefficient $\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B}$. If ρ_{AB} is close to one in absolute value then returns mimic each other extremely closely whereas if ρ_{AB} is close to zero then the returns may show very little relationship.

The portfolio problem is set-up as follows. We have a given amount of wealth and it is assumed that we will exhaust all of our wealth between investments in the two stocks. The investor's problem is to decide how much wealth to put in asset A and how much to put in asset B. Let x_A denote the share of wealth invested in stock A and x_B denote the share of wealth invested in stock B. The values of x_A and x_B can be positive or negative. Positive values denote *long* positions (purchases) in the assets. Negative values denote *short* positions (sales)[1]. Since all wealth is put into the two investments it follows that $x_A + x_B = 1$. Note, if asset A is shorted it is assumed that the proceeds of the short sale are used to purchase asset B. The investor must choose the values of x_A and x_B; that is, how much to invest in asset A and how much to invest in asset B.

Our investment in the two stocks forms a *portfolio* and the shares x_A and x_B are referred to as *portfolio shares* or weights. The return on the portfolio over the next month is a random variable and is given by

$$R_p = x_A R_A + x_B R_B, \tag{1}$$

which is just a simple linear combination or weighted average of the random return variables R_A and R_B. Since R_A and R_B are assumed to be normally distributed, R_p is also normally distributed.

1.1 Portfolio expected return and variance

The return on a portfolio is a random variable and has a probability distribution that depends on the distributions of the assets in the portfolio. However, we can easily deduce some of the properties of this distribution by using the following results concerning linear combinations of random variables:

$$\mu_p = E[R_p] = x_A \mu_A + x_B \mu_B \tag{2}$$

$$\sigma_p^2 = var(R_p) = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \tag{3}$$

These results are so important to portfolio theory that it is worthwhile to go through the derivations. For the first result (2), we have

$$E[R_p] = E[x_A R_A + x_B R_B] = x_A E[R_A] + x_B E[R_B] = x_A \mu_A + x_B \mu_B$$

[1] To short an asset one borrows the asset, usually from a broker, and then sells it. The proceeds from the short sale are usually kept on account with a broker and there often restrictions that prevent the use of these funds for the purchase of other assets. The short position is closed out when the asset is repurchased and then returned to original owner. If the asset drops in value then a gain is made on the short sale and if the asset increases in value a loss is made.

by the linearity of the expectation operator. For the second result (3), we have

$$
\begin{aligned}
var(R_p) &= var(x_A R_A + x_B R_B) = E[(x_A R_A + x_B R_B) - E[x_A R_A + x_B R_B])^2] \\
&= E[(x_A(R_A - \mu_A) + x_B(R_B - \mu_B))^2] \\
&= E[x_A^2(R_A - \mu_A)^2 + x_B^2(R_B - \mu_B)^2 + 2x_A x_B(R_A - \mu_A)(R_B - \mu_B)] \\
&= x_A^2 E[(R_A - \mu_A)^2] + x_B^2 E[(R_B - \mu_B)^2] + 2x_A x_B E[(R_A - \mu_A)(R_B - \mu_B)],
\end{aligned}
$$

and the result follows by the definitions of $var(R_A), var(R_B)$ and $cov(R_A, R_B)$..

Notice that the variance of the portfolio is a weighted average of the variances of the individual assets plus two times the product of the portfolio weights times the covariance between the assets. If the portfolio weights are both positive then a positive covariance will tend to increase the portfolio variance, because both returns tend to move in the same direction, and a negative covariance will tend to reduce the portfolio variance. Thus finding negatively correlated returns can be very beneficial when forming portfolios. What is surprising is that a positive covariance can also be beneficial to diversification.

2 Efficient portfolios with two risky assets

In this section we describe how mean-variance efficient portfolios are constructed. First we make some assumptions:

Assumptions

- Returns are jointly normally distributed over the investment horizon. This implies that means, variances and covariances of returns completely characterize the joint distribution of returns.
- Investors know the values of asset return means, variances and covariances.
- Investors only care about portfolio expected return and portfolio variance. Investors like portfolios with high expected return but dislike portfolios with high return variance.

Given the above assumptions we set out to characterize the set of *efficient portfolios*: those portfolios that have the highest expected return for a given level of risk as measured by portfolio variance. These are the portfolios that investors are most interested in holding.

For illustrative purposes we will show calculations using the data in the table below.

Table 1: Example Data

μ_A	μ_B	σ_A^2	σ_B^2	σ_A	σ_B	σ_{AB}	ρ_{AB}
0.175	0.055	0.067	0.013	0.258	0.115	-0.004875	-0.164

The collection of all *feasible* portfolios, or the *investment possibilities set*, in the case of two assets is simply all possible portfolios that can be formed by varying the portfolio weights x_A and x_B such that the weights sum to one ($x_A + x_B = 1$). We summarize the expected return-risk (mean-variance) properties of the feasible portfolios in a plot with portfolio expected return, μ_p, on the vertical axis and portfolio standard deviation, σ_p, on the horizontal axis. The portfolio standard deviation is used instead of variance because standard deviation is measured in the same units as the expected value (recall, variance is the average squared deviation from the mean).

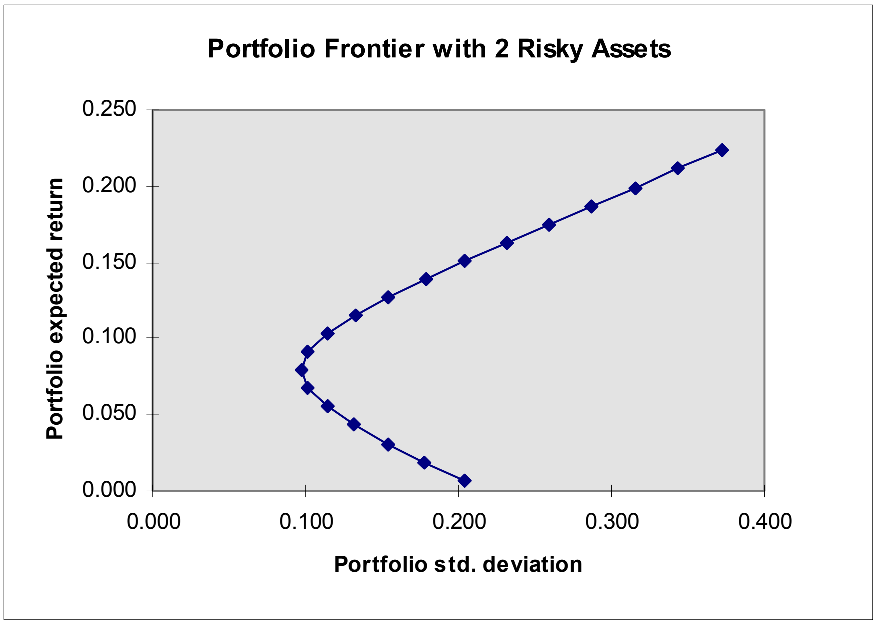


Figure 1

The investment possibilities set or portfolio frontier for the data in Table 1 is illustrated in Figure 1. Here the portfolio weight on asset A, x_A, is varied from -0.4 to 1.4 in increments of 0.1 and, since $x_B = 1 - x_A$, the weight on asset is then varies from 1.4 to -0.4. This gives us 18 portfolios with weights $(x_A, x_B) = (-0.4, 1.4), (-0.3, 1.3), ..., (1.3, -0.3), (1.4, -0.4)$. For each of these portfolios we use the formulas (2) and (3) to compute μ_p and $\sigma_p = \sqrt{\sigma_p^2}$. We then plot these values.

Notice that the plot in (μ_p, σ_p) space looks like a parabola turned on its side (in fact it is one side of a hyperbola). Since investors desire portfolios with the highest expected return, μ_p, for a given level of risk, σ_p, combinations that are in the upper left corner are the best portfolios and those in the lower right corner are the worst. Notice that the portfolio at the bottom of the parabola has the property that it has the smallest variance among all feasible portfolios. Accordingly, this portfolio is called the *global minimum variance* portfolio.

Efficient portfolios are those with the highest expected return for a given level of risk. *Inefficient portfolios* are then portfolios such that there is another feasible portfolio that has the same risk (σ_p) but a higher expected return (μ_p). From the plot it is clear that the inefficient portfolios are the feasible portfolios that lie below the global minimum variance portfolio and the efficient portfolios are those that lie above the global minimum variance portfolio.

2.1 Computing the Global Minimum Variance Portfolio

It is a simple exercise in calculus to find the global minimum variance portfolio. We solve the constrained optimization problem[2]

$$\begin{aligned} \min_{x_A, x_B} \sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_{AB} \\ s.t. \;\; x_A + x_B &= 1. \end{aligned}$$

Substituting $x_B = 1 - x_A$ into the formula for σ_p^2 reduces the problem to

$$\min_{x_A} \sigma_p^2 = x_A^2\sigma_A^2 + (1-x_A)^2\sigma_B^2 + 2x_A(1-x_A)\sigma_{AB}.$$

The first order conditions for a minimum, via the chain rule, are

$$0 = \frac{d\sigma_p^2}{dx_A} = 2x_A^{\min}\sigma_A^2 - 2(1-x_A^{\min})\sigma_B^2 + 2\sigma_{AB}(1-2x_A^{\min})$$

and straightforward calculations yield

$$x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}, \; x_B^{\min} = 1 - x_A^{\min}. \tag{4}$$

For our example, using the data in table 1, we get $x_A^{\min} = 0.2$ and $x_B^{\min} = 0.8$.

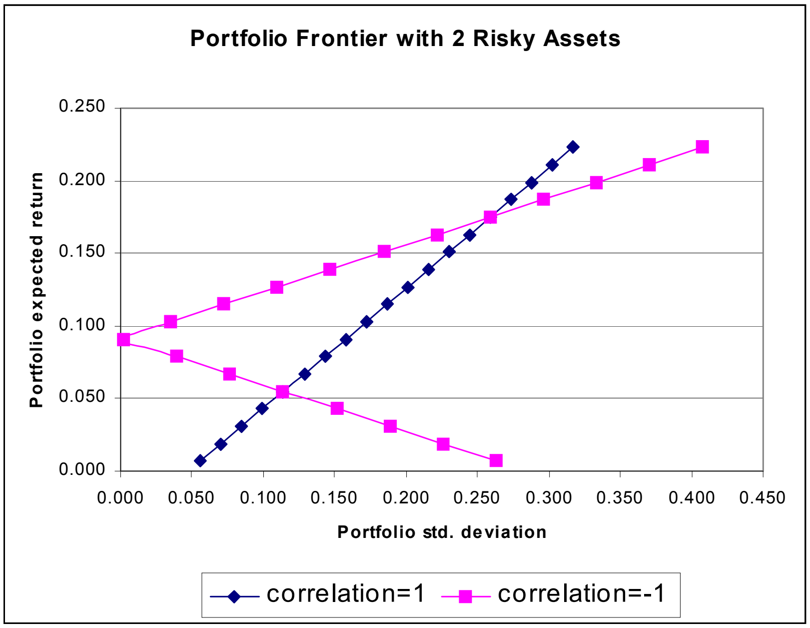
2.2 Correlation and the Shape of the Efficient Frontier

The shape of the investment possibilities set is very sensitive to the correlation between assets A and B. If ρ_{AB} is close to 1 then the investment set approaches a straight line connecting the portfolio with all wealth invested in asset B, $(x_A, x_B) = (0, 1)$, to the portfolio with all wealth invested in asset A, $(x_A, x_B) = (1, 0)$. This case is illustrated in Figure 2. As ρ_{AB} approaches zero the set starts to bow toward the μ_p axis and the power of diversification starts to kick in. If $\rho_{AB} = -1$ then the set actually touches the μ_p axis. What this means is that if assets A and B are perfectly negatively correlated then there exists a portfolio of A and B that has positive expected return and zero variance! To find the portfolio with $\sigma_p^2 = 0$ when $\rho_{AB} = -1$ we use (4) and the fact that $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B$ to give

$$x_A^{\min} = \frac{\sigma_B}{\sigma_A + \sigma_B}, \; x_B^{\min} = 1 - x_A$$

The case with $\rho_{AB} = -1$ is also illustrated in Figure 2.

[2] A review of optimization and constrained optimization is given in the appendix to this chapter.



Given the efficient set of portfolios, which portfolio will an investor choose? Of the efficient portfolios, investors will choose the one that accords with their risk preferences. Very risk averse investors will choose a portfolio very close to the global minimum variance portfolio and very risk tolerant investors will choose portfolios with large amounts of asset A which may involve short-selling asset B.

3 Efficient portfolios with a risk-free asset

In the preceding section we constructed the efficient set of portfolios in the absence of a risk-free asset. Now we consider what happens when we introduce a risk free asset. In the present context, a risk free asset is equivalent to default-free pure discount bond that matures at the end of the assumed investment horizon. The risk-free rate, r_f, is then the nominal return on the bond. For example, if the investment horizon is one month then the risk-free asset is a 30-day Treasury bill (T-bill) and the risk free rate is the nominal rate of return on the T-bill. If our holdings of the risk free asset is positive then we are “lending money” at the risk-free rate and if our holdings are negative then we are “borrowing” at the risk-free rate.

3.1 Efficient portfolios with one risky asset and one risk free asset

Continuing with our example, consider an investment in asset B and the risk free asset (henceforth referred to as a T-bill) and suppose that $r_f = 0.03$. Since the risk free rate is fixed over the investment horizon it has some special properties, namely

$$\begin{aligned} \mu_f &= E[r_f] = r_f \\ var(r_f) &= 0 \\ cov(R_B, r_f) &= 0 \end{aligned}$$

Let x_B denote the share of wealth in asset B and $x_f = 1 - x_B$ denote the share of wealth in T-bills. The portfolio expected return is

$$\begin{aligned} R_p &= x_B R_B + (1 - x_B) r_f \\ &= x_B(R_B - r_f) + r_f \end{aligned}$$

The quantity $R_B - r_f$ is called the *excess return* (over the return on T-bills) on asset B. The portfolio expected return is then

$$\mu_p = x_B(\mu_B - r_f) + r_t$$

where the quantity $(\mu_B - r_f)$ is called the *expected excess return* or *risk premium* on asset B. We may express the risk premium on the portfolio in terms of the risk premium on asset B:

$$\mu_p - r_f = x_B(\mu_B - r_f)$$

The more we invest in asset B the higher the risk premium on the portfolio.

The portfolio variance only depends on the variability of asset B and is given by

$$\sigma_p^2 = x_B^2 \sigma_B^2.$$

The portfolio standard deviation is therefore proportional to the standard deviation on asset B:

$$\sigma_p = x_B \sigma_B$$

which can use to solve for x_B

$$x_B = \frac{\sigma_p}{\sigma_B}$$

Using the last result, the feasible (and efficient) set of portfolios follows the equation

$$\mu_p = r_f + \frac{\mu_B - r_f}{\sigma_B} \cdot \sigma_p \tag{5}$$

which is simply straight line in (μ_p, σ_p) with intercept r_f and slope $\frac{\mu_B - r_f}{\sigma_B}$. The slope of the combination line between T-bills and a risky asset is called the *Sharpe ratio* or *Sharpe's slope* and it measures the risk premium on the asset per unit of risk (as measured by the standard deviation of the asset).

The portfolios which are combinations of asset A and T-bills and combinations of asset B and T-bills using the data in Table 1 with $r_f = 0.03$. is illustrated in Figure 4.

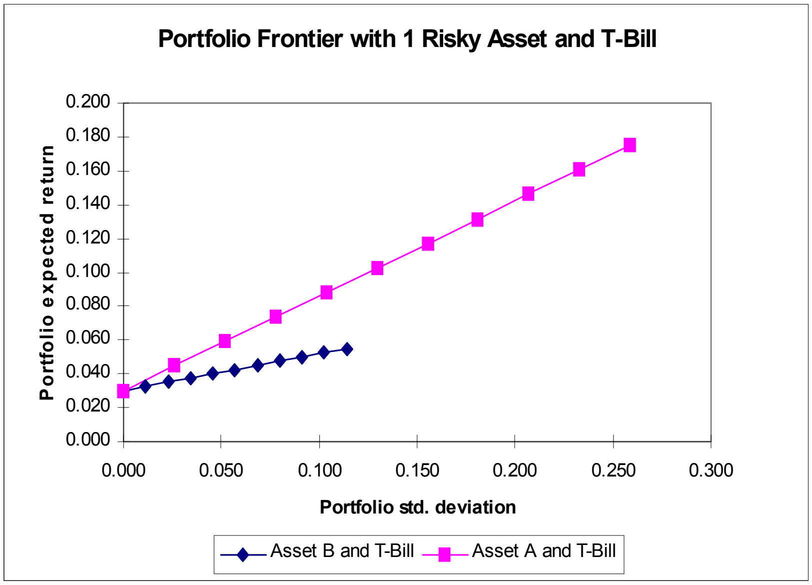


Figure 3

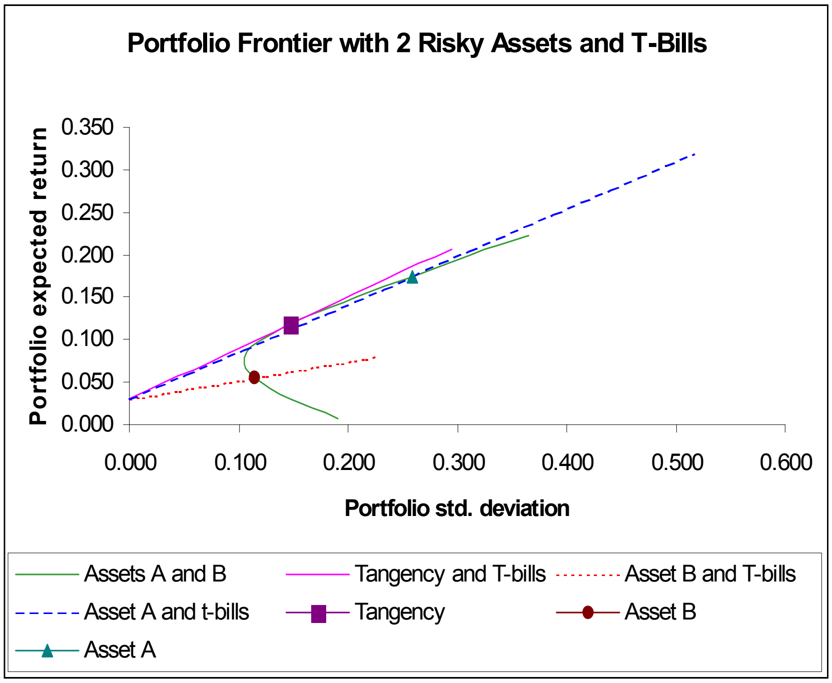
Notice that expected return-risk trade off of these portfolios is linear. Also, notice that the portfolios which are combinations of asset A and T-bills have expected returns uniformly higher than the portfolios consisting of asset B and T-bills. This occurs because the Sharpe's slope for asset A is higher than the slope for asset B:

$$\frac{\mu_A - r_f}{\sigma_A} = \frac{0.175 - 0.03}{0.258} = 0.562, \quad \frac{\mu_B - r_f}{\sigma_B} = \frac{0.055 - 0.03}{0.115} = 0.217.$$

Hence, portfolios of asset A and T-bills are efficient relative to portfolios of asset B and T-bills.

4 Efficient portfolios with two risky assets and a risk-free asset

Now we expand on the previous results by allowing our investor to form portfolios of assets A, B and T-bills. The efficient set in this case will still be a straight line in $(\mu_p, \sigma_p)-$ space with intercept r_f. The slope of the efficient set, the maximum Sharpe ratio, is such that it is tangent to the efficient set constructed just using the two risky assets A and B. Figure 5 illustrates why this is so.



If we invest in only in asset B and T-bills then the Sharpe ratio is $\frac{\mu_B - r_f}{\sigma_B} = 0.217$ and the CAL intersects the parabola at point B. This is clearly not the efficient set of portfolios. For example, we could do uniformly better if we instead invest only in asset A and T-bills. This gives us a Sharpe ratio of $\frac{\mu_A - r_f}{\sigma_A} = 0.562$ and the new CAL intersects the parabola at point A. However, we could do better still if we invest in T-bills and some combination of assets A and B. Geometrically, it is easy to see that the best we can do is obtained for the combination of assets A and B such that the CAL is just tangent to the parabola. This point is marked T on the graph and represents the *tangency portfolio* of assets A and B.

4.1 Solving for the Tangency Portfolio

We can determine the proportions of each asset in the tangency portfolio by finding the values of x_A and x_B that maximize the Sharpe ratio of a portfolio that is on the envelope of the parabola. Formally, we solve the constrained maximization

$$\max_{x_A, x_B} \frac{\mu_p - r_f}{\sigma_p} \quad s.t.$$

$$\begin{aligned} \mu_p &= x_A \mu_A + x_B \mu_B \\ \sigma_p^2 &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \\ 1 &= x_A + x_B \end{aligned}$$

After various substitutions, the above problem can be reduced to

$$\max_{x_A} \frac{x_A(\mu_A - r_f) + (1 - x_A)(\mu_B - r_f)}{\left(x_A^2\sigma_A^2 + (1 - x_A)^2\sigma_B^2 + 2x_A(1 - x_A)\sigma_{AB}\right)^{1/2}}.$$

This is a straightforward, albeit very tedious, calculus problem and the solution can be shown to be

$$x_A^T = \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}, \; x_B^T = 1 - x_A^T.$$

For the example data using $r_f = 0.03$, we get $x_A^T = 0.542$ and $x_B^T = 0.458$. The expected return on the tangency portfolio is

$$\begin{aligned} \mu_T &= x_A^T\mu_A + x_B^T\mu_B \\ &= (0.542)(0.175) + (0.458)(0.055) = 0.110, \end{aligned}$$

the variance of the tangency portfolio is

$$\begin{aligned} \sigma_T^2 &= \left(x_A^T\right)^2\sigma_A^2 + \left(x_B^T\right)^2\sigma_B^2 + 2x_A^Tx_B^T\sigma_{AB} \\ &= (0.542)^2(0.067) + (0.458)^2(0.013) + 2(0.542)(0.458) = 0.015, \end{aligned}$$

and the standard deviation of the tangency portfolio is

$$\sigma_T = \sqrt{\sigma_T^2} = \sqrt{0.015} = 0.124.$$

4.2 Efficient Portfolios of Two Risky Assets and T-Bill

The efficient portfolios now are combinations of the tangency portfolio and the T-bill. This important result is known as the *mutual fund separation theorem*. The tangency portfolio can be considered as a mutual fund of the two risky assets, where the shares of the two assets in the mutual fund are determined by the tangency portfolio weights, and the T-bill can be considered as a mutual fund of risk free assets. The expected return-risk trade-off of these portfolios is given by the line connecting the risk-free rate to the tangency point on the efficient frontier of risky asset only portfolios. Which combination of the tangency portfolio and the T-bill an investor will choose depends on the investor's risk preferences. If the investor is very risk averse, then she will choose a combination with very little weight in the tangency portfolio and a lot of weight in the T-bill. This will produce a portfolio with an expected return close to the risk free rate and a variance that is close to zero.

For example, a highly risk averse investor may choose to put 10% of her wealth in the tangency portfolio and 90% in the T-bill. Then she will hold $(10\%) \times (54.2\%) =$

5.42% of her wealth in asset A, $(10\%) \times (45.8\%) = 4.58\%$ of her wealth in asset B and 90% of her wealth in the T-bill. The expected return on this portfolio is

$$\begin{aligned}
\mu_p &= r_f + 0.10(\mu_T - r_f) \\
&= 0.03 + 0.10(0.110 - 0.03) \\
&= 0.038.
\end{aligned}$$

and the standard deviation is

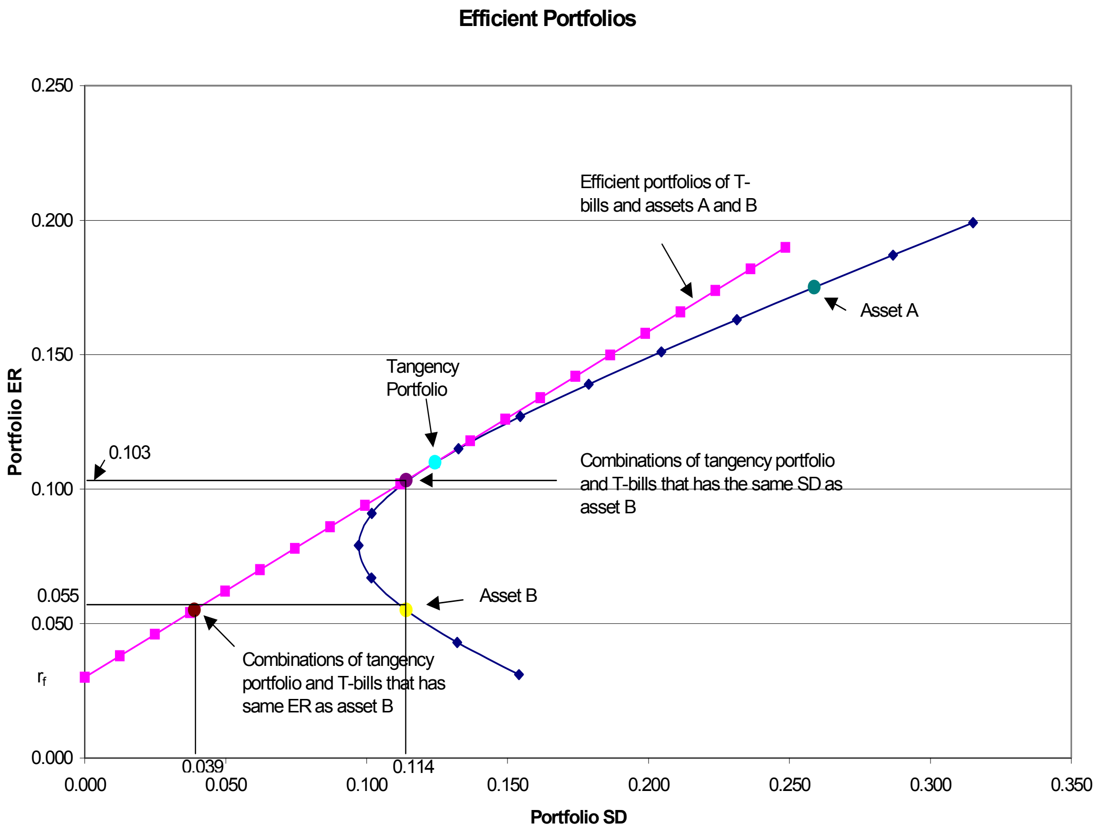
$$\begin{aligned}
\sigma_p &= 0.10\sigma_T \\
&= 0.10(0.124) \\
&= 0.012.
\end{aligned}$$

A very risk tolerant investor may actually borrow at the risk free rate and use these funds to leverage her investment in the tangency portfolio. For example, suppose the risk tolerant investor borrows 10% of her wealth at the risk free rate and uses the proceed to purchase 110% of her wealth in the tangency portfolio. Then she would hold $(110\%) \times (54.2\%) = 59.62\%$ of her wealth in asset A, $(110\%) \times (45.8\%) = 50.38\%$ in asset B and she would owe 10% of her wealth to her lender. The expected return and standard deviation on this portfolio is

$$\begin{aligned}
\mu_p &= 0.03 + 1.1(0.110 - 0.03) = 0.118 \\
\sigma_p &= 1.1(0.124) = 0.136.
\end{aligned}$$

4.3 Interpreting Efficient Portfolios

As we have seen, efficient portfolios are those portfolios that have the highest expected return for a given level of risk as measured by portfolio standard deviation. For portfolios with expected returns above the T-bill rate, efficient portfolios can also be characterized as those portfolios that have minimum risk (as measured by portfolio standard deviation) for a given target expected return.



To illustrate, consider figure 5 which shows the portfolio frontier for two risky assets and the efficient frontier for two risky assets plus a risk-free asset. Suppose an investor initially holds all of his wealth in asset A. The expected return on this portfolio is $\mu_B = 0.055$ and the standard deviation (risk) is $\sigma_B = 0.115$. An efficient portfolio (combinations of the tangency portfolio and T-bills) that has the same standard deviation (risk) as asset B is given by the portfolio on the efficient frontier that is directly above $\sigma_B = 0.115$. To find the shares in the tangency portfolio and T-bills in this portfolio recall from (xx) that the standard deviation of a portfolio with x_T invested in the tangency portfolio and $1 - x_T$ invested in T-bills is $\sigma_p = x_T \sigma_T$. Since we want to find the efficient portfolio with $\sigma_p = \sigma_B = 0.115$, we solve

$$x_T = \frac{\sigma_B}{\sigma_T} = \frac{0.115}{0.124} = 0.917, \;\; x_f = 1 - x_T = 0.083.$$

That is, if we invest 91.7% of our wealth in the tangency portfolio and 8.3% in T-bills we will have a portfolio with the same standard deviation as asset B. Since this is an efficient portfolio, the expected return should be higher than the expected return on asset B. Indeed it is since

$$\begin{aligned}\mu_p &= r_f + x_T(\mu_T - r_f) \\ &= 0.03 + 0.917(0.110 - 0.03) \\ &= 0.103\end{aligned}$$

Notice that by diversifying our holding into assets A, B and T-bills we can obtain a portfolio with the same risk as asset B but with almost twice the expected return!

Next, consider finding an efficient portfolio that has the same expected return as asset B. Visually, this involves finding the combination of the tangency portfolio and T-bills that corresponds with the intersection of a horizontal line with intercept $\mu_B = 0.055$ and the line representing efficient combinations of T-bills and the tangency portfolio. To find the shares in the tangency portfolio and T-bills in this portfolio recall from (xx) that the expected return of a portfolio with x_T invested in the tangency portfolio and $1 - x_T$ invested in T-bills has expected return equal to $\mu_p = r_f + x_T(\mu_T - r_f)$. Since we want to find the efficient portfolio with $\mu_p = \mu_B = 0.055$ we use the relation

$$\mu_p - r_f = x_T(\mu_T - r_F)$$

and solve for x_T and $x_f = 1 - x_T$

$$x_T = \frac{\mu_p - r_f}{\mu_T - r_f} = \frac{0.055 - 0.03}{0.110 - 0.03} = 0.313, x_f = 1 - x_T = 0.687.$$

That is, if we invest 31.3% of wealth in the tangency portfolio and 68.7% of our wealth in T-bills we have a portfolio with the same expected return as asset B. Since this is an efficient portfolio, the standard deviation (risk) of this portfolio should be lower than the standard deviation on asset B. Indeed it is since

$$\begin{aligned}\sigma_p &= x_T \sigma_T \\ &= 0.313(0.124) \\ &= 0.039.\end{aligned}$$

Notice how large the risk reduction is by forming an efficient portfolio. The standard deviation on the efficient portfolio is almost three times smaller than the standard deviation of asset B!

The above example illustrates two ways to interpret the benefits from forming efficient portfolios. Starting from some benchmark portfolio, we can fix standard deviation (risk) at the value for the benchmark and then determine the gain in expected return from forming a diversified portfolio[3]. The gain in expected return has concrete

[3]The gain in expected return by investing in an efficient portfolio abstracts from the costs associated with selling the benchmark portfolio and buying the efficient portfolio.

meaning. Alternatively, we can fix expected return at the value for the benchmark and then determine the reduction in standard deviation (risk) from forming a diversified portfolio. The meaning to an investor of the reduction in standard deviation is not as clear as the meaning to an investor of the increase in expected return. It would be helpful if the risk reduction benefit can be translated into a number that is more interpretable than the standard deviation. The concept of Value-at-Risk (VaR) provides such a translation.

4.4 Efficient Portfolios and Value-at-Risk

Recall, the VaR of an investment is the expected loss in investment value over a given horizon with a stated probability. For example, consider an investor who invests $W_0 = \$100,000$ in asset B over the next year. Assume that R_B represents the annual (continuously compounded) return on asset B and that $R_B \,\tilde{}\, N(0.055, (0.114)^2)$. The 5% annual VaR of this investment is the loss that would occur if return on asset B is equal to the 5% left tail quantile of the normal distribution of R_B. The 5% quantile, $q_{0.05}$ is determined by solving

$$\Pr(R_B \leq q_{0.05}) = 0.05.$$

Using the inverse cdf for a normal random variable with mean 0.055 and standard deviation 0.114 it can be shown that $q_{0.05} = -0.133$.That is, with 5% probability the return on asset B will be -13.3% or less. If $R_B = -0.133$ then the loss in portfolio value[4], which is the 5% VaR, is

$$\text{loss in portfolio value} = VaR = |W_0 \cdot (e^{q_{0.05}} - 1)| = |\$100,000(e^{-0.133} - 1)| = \$12,413.$$

To reiterate, if the investor hold \$100,000 in asset B over the next year then the 5% VaR on the portfolio is $\$12,413$. This is the loss that would occur with 5% probability.

Now suppose the investor chooses to hold an efficient portfolio with the same expected return as asset B. This portfolio consists of 31.3% in the tangency portfolio and 68.7% in T-bills and has a standard deviation equal to 0.039. Let R_p denote the annual return on this portfolio and assume that $R_p \,\tilde{}\, N(0.055, 0.039)$. Using the inverse cdf for this normal distribution, the 5% quantile can be shown to be $q_{0.05} = -0.009$. That is, with 5% probability the return on the efficient portfolio will be -0.9% or less. This is considerably smaller than the 5% quantile of the distribution of asset B. If $R_p = -0.009$ the loss in portfolio value (5% VaR) is

$$\text{loss in portfolio value} = VaR = |W_0 \cdot (e^{q_{0.05}} - 1)| = |\$100,000(e^{-0.009} - 1)| = \$892.$$

Notice that the 5% VaR for the efficient portfolio is almost fifteen times smaller than the 5% VaR of the investment in asset B. Since VaR translates risk into a dollar figure it is more interpretable than standard deviation.

[4] To compute the VaR we need to convert the continuous compounded return (quantile) to a simple return (quantile). Recall, if R_t^c is a continuously compounded return and R_t is a somple return then $R_t^c = \ln(1 + R_t)$ and $R_t = e^{R_t^c} - 1$.

5 Statistical Analysis of Efficient Portfolios

- Discuss practical implementation of portfolio theory - have to estimate means, variances and covariances
- Use simulation analysis to evaluate how well estimated optimal portfolios relate to actual optimal portfolios. Here the optimal portfolio weights are not known and must be estimated. The formula $\mu_p - r_f = x_T(\mu_T - r_F)$ with $x_T = \sigma_p/\sigma_T$ needs to be estimated from actual data. How good of an estimator of this quantity do we actually get? Jobson and Korke say this this is estimated very badly.
- Use simulation analysis to evaluate how well estimated optimal portfolios perform relative to actual optimal portfolios.
- Use simulation analysis to show the variation in the frontier. Calibrate the simulation to actual data. My guess is that the frontier will wabble around considerably.

6 Further Reading

The classic text on portfolio optimization is Markowitz (1954). Good intermediate level treatments are given in Benninga (2000), Bodie, Kane and Marcus (1999) and Elton and Gruber (1995). An interesting recent treatment with an emphasis on statistical properties is Michaud (1998). Many practical results can be found in the *Financial Analysts Journal* and the *Journal of Portfolio Management*. An excellent overview of value at risk is given in Jorian (1997).

7 Appendix Review of Optimization and Constrained Optimization

Consider the function of a single variable

$$y = f(x) = x^2$$

which is illustrated in Figure xxx. Clearly the minimum of this function occurs at the point $x = 0$. Using calculus, we find the minimum by solving

$$\min_x \; y = x^2.$$

The first order (necessary) condition for a minimum is

$$0 = \frac{d}{dx}f(x) = \frac{d}{dx}x^2 = 2x$$

and solving for x gives $x = 0$. The second order condition for a minimum is

$$0 < \frac{d^2}{dx} f(x)$$

and this condition is clearly satisfied for $f(x) = x^2$.

Next, consider the function of two variables

$$y = f(x, z) = x^2 + z^2 \tag{6}$$

which is illustrated in Figure xxx.

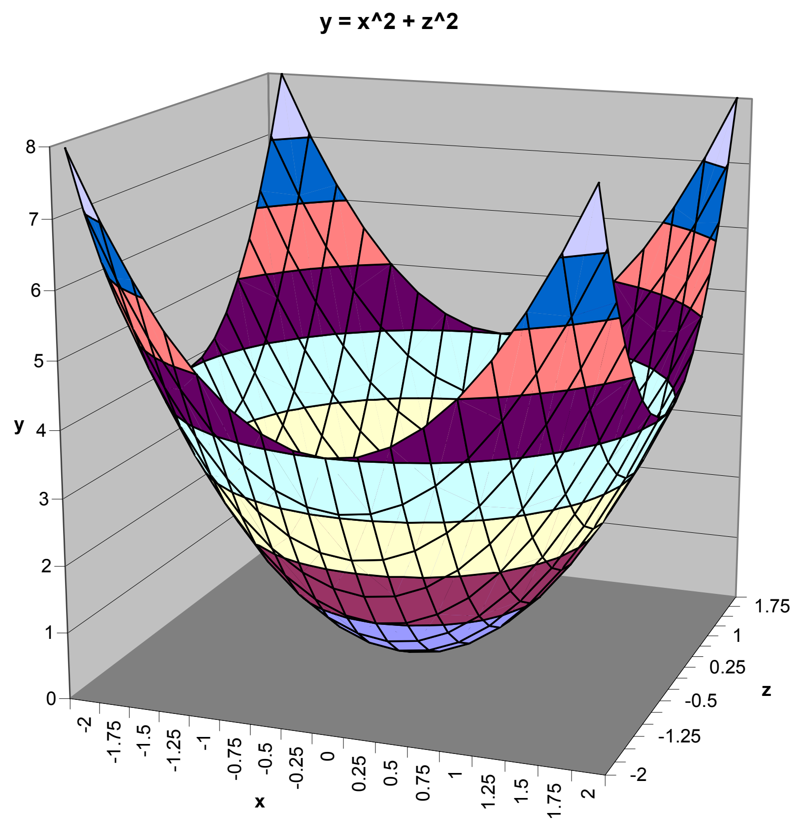


Figure 6

This function looks like a salad bowl whose bottom is at $x = 0$ and $z = 0$. To find the minimum of (6), we solve

$$\min_{x,z} y = x^2 + z^2$$

and the first order necessary conditions are

$$0 = \frac{\partial y}{\partial x} = 2x$$

and

$$0 = \frac{\partial y}{\partial z} = 2z.$$

Solving these two equations gives $x = 0$ and $z = 0$.

Now suppose we want to minimize (6) subject to the linear constraint

$$x + z = 1. \tag{7}$$

The minimization problem is now a *constrained minimization*

$$\begin{aligned} \min_{x,z} y &= x^2 + z^2 \text{ subject to } (s.t.) \\ x + z &= 1 \end{aligned}$$

and is illustrated in Figure xxx. Given the constraint $x + z = 1$, the function (6) is no longer minimized at the point $(x, z) = (0, 0)$ because this point does not satisfy $x + z = 1$. The One simple way to solve this problem is to substitute the restriction (7) into the function (6) and reduce the problem to a minimization over one variable. To illustrate, use the restriction (7) to solve for z as

$$z = 1 - x. \tag{8}$$

Now substitute (7) into (6) giving

$$y = f(x, z) = f(x, 1 - x) = x^2 + (1 - x)^2. \tag{9}$$

The function (9) satisfies the restriction (7) by construction. The constrained minimization problem now becomes

$$\min_x y = x^2 + (1 - x)^2.$$

The first order conditions for a minimum are

$$0 = \frac{d}{dx}(x^2 + (1 - x)^2) = 2x - 2(1 - x) = 4x - 2$$

and solving for x gives $x = 1/2$. To solve for z, use (8) to give $z = 1 - (1/2) = 1/2$. Hence, the solution to the constrained minimization problem is $(x, z) = (1/2, 1/2)$.

Another way to solve the constrained minimization is to use the method of *Lagrange multipliers*. This method augments the function to be minimized with a linear function of the constraint in homogeneous form. The constraint (7) in homogenous form is

$$x + z - 1 = 0$$

The augmented function to be minimized is called the *Lagrangian* and is given by

$$L(x, z, \lambda) = x^2 + z^2 - \lambda(x + z - 1).$$

The coefficient on the constraint in homogeneous form, λ, is called the Lagrange multiplier. It measures the cost, or shadow price, of imposing the constraint relative to the unconstrained problem. The constrained minimization problem to be solved is now

$$\min_{x,z,\lambda} L(x,z,\lambda) = x^2 + z^2 + \lambda(x + z - 1).$$

The first order conditions for a minimum are

$$\begin{aligned} 0 &= \frac{\partial L(x,z,\lambda)}{\partial x} = 2x + \lambda \\ 0 &= \frac{\partial L(x,z,\lambda)}{\partial z} = 2z + \lambda \\ 0 &= \frac{\partial L(x,z,\lambda)}{\partial \lambda} = x + z - 1 \end{aligned}$$

The first order conditions give three linear equations in three unknowns. Notice that the first order condition with respect to λ imposes the constraint. The first two conditions give

$$2x = 2z = -\lambda$$

or

$$x = z.$$

Substituting $x = z$ into the third condition gives

$$2z - 1 = 0$$

or

$$z = 1/2.$$

The final solution is $(x, y, \lambda) = (1/2, 1/2, -1)$.

The Lagrange multiplier, λ, measures the marginal cost, in terms of the value of the objective function, of imposing the constraint. Here, $\lambda = -1$ which indicates that imposing the constraint $x + z = 1$ reduces the objective function. To understand the roll of the Lagrange multiplier better, consider imposing the constraint $x + z = 0$. Notice that the unconstrained minimum achieved at $x = 0, z = 0$ satisfies this constraint. Hence, imposing $x + z = 0$ does not cost anything and so the Lagrange multiplier associated with this constraint should be zero. To confirm this, the we solve the problem

$$\min_{x,z,\lambda} L(x,z,\lambda) = x^2 + z^2 + \lambda(x + z - 0).$$

The first order conditions for a minimum are

$$\begin{aligned} 0 &= \frac{\partial L(x,z,\lambda)}{\partial x} = 2x - \lambda \\ 0 &= \frac{\partial L(x,z,\lambda)}{\partial z} = 2z - \lambda \\ 0 &= \frac{\partial L(x,z,\lambda)}{\partial \lambda} = x + z \end{aligned}$$

The first two conditions give

$$2x = 2z = -\lambda$$

or

$$x = z.$$

Substituting $x = z$ into the third condition gives

$$2z = 0$$

or

$$z = 0.$$

The final solution is $(x, y, \lambda) = (0, 0, 0)$. Notice that the Lagrange multiplier, λ, is equal to zero in this case.

8 Problems

Exercise 1 *Consider the problem of investing in two risky assets A and B and a risk-free asset (T-bill). The optimization problem to find the tangency portfolio may be reduced to*

$$\max_{x_A} \frac{x_A(\mu_A - r_f) + (1 - x_A)(\mu_B - r_f)}{(x_A^2\sigma_A^2 + (1 - x_A)^2\sigma_B^2 + 2x_A(1 - x_A)\sigma_{AB})^{1/2}}$$

where x_A is the share of wealth in asset A in the tangency portfolio and $x_B = 1 - x_A$ is the share of wealth in asset B in the tangency portfolio. Using simple calculus, show that

$$x_A = \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}.$$

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