FINC 748: Investments	**Ramana Sonti**
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Lecture Note 4A: The Capital Asset Pricing Model

To be read with BKM Chapter 9

- Assumptions of the CAPM
- Proof of the CAPM
- Intuition behind the CAPM
- The Security Market Line
- The certainty equivalent form of the CAPM

- **Introduction**

  - Markowitz portfolio theory provides us with tools to select optimal portfolios *given* expected returns, variances and covariances. Where do we get these inputs? It turns out that using historical variances and covariances is a reasonable starting point, but simply taking the historical average return as a proxy for *expected return* is especially troublesome. We need *Asset Pricing Models* to tell us how expected returns are *economically* determined.
  - Asset Pricing Models could be *theoretical* (derived rigorously from fundamental principles), or *empirical* (based on observed relationships in historical data on asset returns).
  - In this lecture, we shall study one such Asset Pricing Model, the so-called Capital Asset Pricing Model (hereafter, the CAPM). The CAPM is:
    a) a theoretical APM.
    b) The earliest APM of modern finance
    c) An equilibrium model, which specifies the relationship between *expected return* and *systematic risk*.
  - What is equilibrium?
    ✓ In economics, *equilibrium* characterizes a situation where no investor wants to do anything differently.

  - <u>A bit of bad news in advance</u>:

    The CAPM does not seem to hold up in *historical* data i.e. asset prices do not seem to follow the relationship posited by the CAPM. (Much more on this later.) Then, you may ask, why do we study it?

    Because, among other things, it gives us some nice intuition in order to understand more advanced and complicated APMs.

- **Assumptions underlying the CAPM**

1. Investors are in *perfect competition* with each other. In our context, this means that no one investor has the ability to influence prices with her/his trade.
2. All investors have the same one-period horizon. The theory does not say whether this period is one year, one month or one week.
3. All investors are Markowitz efficient investors, who care only about the mean and variance (standard deviation) of their portfolios.
4. All investors have homogeneous expectations i.e. have identical probability distributions regarding rate of return. This means all investors assume the same $E(\tilde{r})$ and $\sigma(\tilde{r})$ for different securities, and identical covariances between pairs of securities.
5. Markets are frictionless. This is a collection of assumptions that includes:
    - Investors can borrow or lend any amount at the risk-free rate
    - All investments are infinitely divisible, which means one can buy or sell any fraction of any asset.
    - No taxes or transactions costs
    - There is no inflation or change in interest rates, or inflation is fully anticipated.
6. Capital Markets are in equilibrium. That means all securities are properly priced taking into account their risk.

These assumptions are clearly very stringent and nobody is naïve enough to believe that these assumptions are *actually true*. However, the nature of these assumptions is the price we pay for the simple, yet powerful, results of this theory. Many of these assumptions can be, and have been, relaxed. In all extensions, however, the basic intuition remains the same (even though the algebra and calculations get a lot messier).

- **The Market Portfolio**

  - The above assumptions mean that *every* investor solves an identical Markowitz optimization problem, obtains the same mean-variance frontier, and decides to invest in a combination of:

    a) The risk-free asset

    b) The MVE (optimal risky) portfolio

  - The key question to ask is: If every investor holds the *same* risky portfolio, what *must* that portfolio be? The answer to this question is the key to the CAPM. To answer, let's consider the following example.

Example: Consider a highly simplified world with two risky assets, MSFT and INTC, and one risk-free asset. There are only two investors in this world: Mr. Sqeaumish (S), and Ms. Bungee Jumper (BJ), both with $1 million apiece. Their optimal complete portfolios have the following allocations.

Investor	Risk-free	MVE
S	40%	60%
BJ	-40%	140%

Further, suppose the MVE portfolio consists of 70% MSFT, 30% INTC. Let us look at the total investment in MSFT, INTC and the risk-free asset.

Investor	MSFT	INTC	Risk-free	Total
S	420,000	180,000	400,000	**1,000,000**
BJ	980,000	420,000	-400,000	**1,000,000**
Total	**1,400,000**	**600,000**	**0**	**2,000,000**

1) Since there are only two investors in the economy, the amount borrowed by BJ at the risk-free rate must be the amount lent by S at the same rate. Thus *the net supply of risk-free assets is zero*.

2) The total amount of risky securities in the market is \$2,000,000. The proportion of MSFT of this total is \$1,400,000/\$2,000,000 = 70%, while the proportion of INTC of the total is 30%.
3) These proportions are the same as those of MSFT and INTC in the MVE.
4) That means, if you add up the portfolios of all investors, the total portfolio contains each risky asset in the same proportion as in the MVE. *That means the market portfolio is MVE!*
   - ✓ The *market portfolio* is a portfolio of all risky assets in the economy, with each asset in proportion to its market value. i.e. the weight of each asset i in the market portfolio is:

$$w_i = \frac{Market\ capitalization\ of\ asset\ i}{Total\ market\ capitalization\ of\ all\ risky\ assets}$$

5) This means that the only portfolio held by all investors in addition to the risk-free asset is the market portfolio.

- Hence under the CAPM assumptions, we need not actually formulate and solve the Markowitz problem for the optimal (MVE) portfolio of risky assets. We *know* that the optimal portfolio *will be* the market portfolio.
- How can we solve a problem without even knowing what the inputs are? Isn't this the strangest thing? Such is the logic of economic equilibrium.
- Let's say some new information comes out that makes you believe that MSFT is underpriced, so your Markowitz optimization model tells you that there should be a bit more MSFT in *your* MVE portfolio. But, at the exact same moment, every other investor is also doing the same calculation, and everyone will add a bit more of MSFT to *their* MVE portfolios. The aggregate effect of this will be to increase the proportion of MSFT in the *market portfolio*, until the proportions of MSFT and INTC in

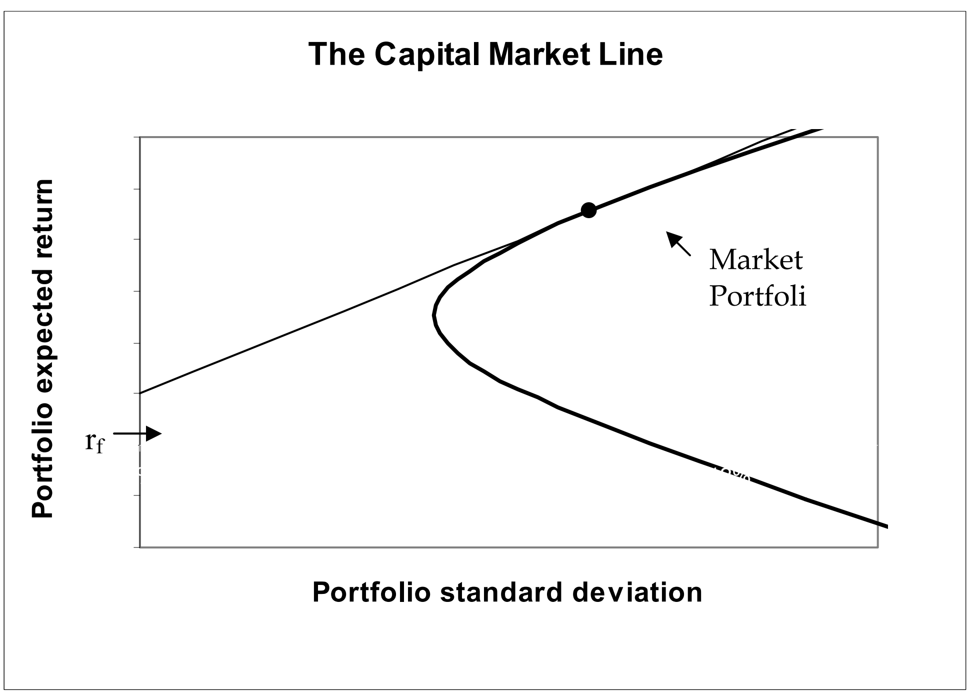
the revised market portfolio are the same as those in the new MVE portfolios. That's equilibrium for you.

- Since the market is in equilibrium *at all times*, one need only apply this logic to conclude that the market portfolio is the MVE portfolio. This suggest that investors do the following:

  Step 1: As a no-brainer, they invest in the MVE portfolio. This is their *investment decision*.

  Step 2: Now, to attain their preferred point along the CAL, they will decide how much to *borrow* or *lend* at the risk-free rate. This is their *financing decision*. This result is named in honor of James Tobin (1981 Nobel prize winner in economics), who first proposed it in 1958, and is called *Tobin's two-fund separation theorem*.

- **The Capital Market Line**
  - The Capital Market Line (CML) is nothing but our old Capital Allocation Line (CAL) with the knowledge that the MVE portfolio is the market portfolio. The CML illustrates two-fund separation theorem nicely.



- All efficient portfolios lie along the CML. Algebraically, this means that every efficient asset or portfolio satisfies the CML equation, which is:

  $E(\tilde{r}_e) = r_f + \left(\frac{E(\tilde{r}_m) - r_f}{\sigma_m}\right).\sigma_e$, where $E(\tilde{r}_m)$ and σ_m are the expected return and standard deviation of the market portfolio, and $E(\tilde{r}_e)$ and σ_e are the corresponding quantities for any arbitrary efficient asset (or portfolio).
- Intuitively, the CML says that if risk is measured by standard deviation, then as risk increases, the corresponding expected rate of return also must increase.
- This is the first part of the CAPM, relating efficient portfolios to the market portfolio. The next part of the CAPM, which might be more familiar to you is concerned with the relationship between expected return of any (not just efficient) asset to its individual risk. This is the *pricing model* part of the CAPM.

- **The Pricing Model**

  ✓ CAPM: The expected return of any asset, $E(\tilde{r}_i)$ satisfies:

  $$E(\tilde{r}_i) = r_f + \beta_i \left[E(\tilde{r}_m) - r_f \right], \text{ where } \beta_i = \frac{Cov(\tilde{r}_i, \tilde{r}_m)}{\sigma_m^2} = \frac{\sigma_{i,m}}{\sigma_m^2}$$

  Proof: I will present here a standard mathematical proof, followed by an intuitive proof.

  Let us say we have $1 worth of the market portfolio *m*, which has an expected return of $E(\tilde{r}_m)$ and a standard deviation σ_m. To this portfolio, consider adding a wee bit of asset *i*. Specifically, consider adding an amount $\$\alpha$ of asset i. For investing this amount, assume we borrow $\$\alpha$ at the risk-free rate. Then, the return on the new portfolio is

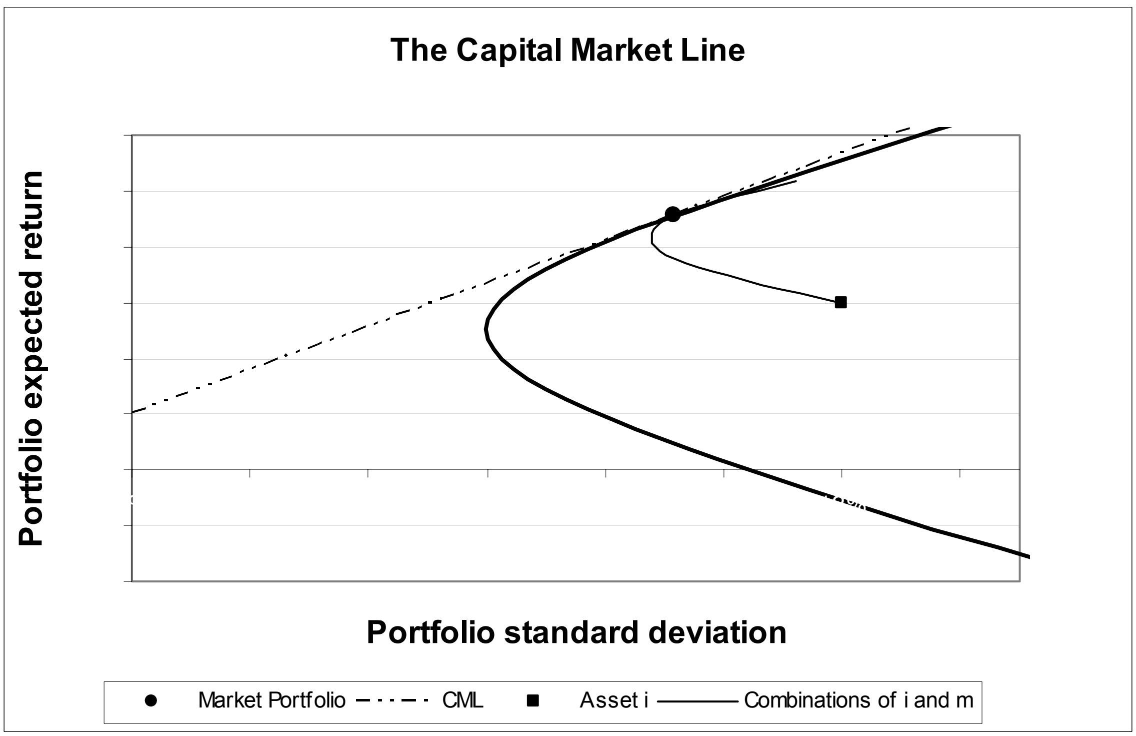
$\tilde{r}_p = \tilde{r}_m + \alpha(\tilde{r}_i - r_f)$. From this, we can determine the expected return and standard deviation of the portfolio as:

$$E(\tilde{r}_p) = E(\tilde{r}_m) + \alpha\left[E(\tilde{r}_i) - r_f\right] \qquad \ldots \qquad \text{(A) , and}$$

$$\sigma_p^2 = Var\left(\tilde{r}_m + \alpha\left[E(\tilde{r}_i) - r_f\right]\right) = \sigma_m^2 + \alpha^2.\sigma_i^2 + 2\alpha\sigma_{i,m}$$

$$\Rightarrow \sigma_p = \left[\sigma_m^2 + \alpha^2.\sigma_i^2 + 2\alpha\sigma_{i,m}\right]^{1/2} \qquad \ldots \qquad \text{(B)}$$

As α varies, all possible combinations of the asset and the market trace a curve, which lies completely inside the CML. This curve can never cross the CML (Why?). See the following figure for a visual representation of these portfolio combinations.



Since α = 0 is simply the market portfolio, at α =0, this curve must be tangent to the CML at the market portfolio point. This tangency is what we exploit to derive the CAPM pricing formula.

Tangency means that the slope of this "combinations" curve should be equal to the slope of the CML at α = 0. Let us find both slopes algebraically and set them equal to each other.

Differentiating equation (A) w.r.t. α, we have: $\frac{dE(\tilde{r}_p)}{d\alpha} = E(\tilde{r}_i) - r_f$ … (C)

Using (B), we have $\frac{d\sigma_p}{d\alpha} = \frac{1}{\sigma_p}\left[\alpha.\sigma_i^2 + \sigma_{i,m}\right]$ … (D)

Using the results in equations (C) and (D), we have:

$$\frac{dE(\tilde{r}_p)}{d\sigma_p} = \frac{\frac{dE(\tilde{r}_p)}{d\alpha}}{\frac{d\sigma_p}{d\alpha}} = \frac{E(\tilde{r}_i) - r_f}{\frac{1}{\sigma_p}\left[\alpha.\sigma_i^2 + \sigma_{i,m}\right]}$$

$$\Rightarrow \left.\frac{dE(\tilde{r}_p)}{d\sigma_p}\right|_{\alpha=0} = \frac{E(\tilde{r}_i) - r_f}{\frac{1}{\sigma_m}.\sigma_{i,m}} = \frac{\left[E(\tilde{r}_i) - r_f\right].\sigma_m}{\sigma_{i,m}}$$

This is the slope of the "combinations" curve at α = 0. We know that the slope of the CML is $\left(\frac{E(\tilde{r}_m) - r_f}{\sigma_m}\right)$. Let us equate both slopes:

$\frac{\left[E(\tilde{r}_i) - r_f\right].\sigma_m}{\sigma_{i,m}} = \frac{E(\tilde{r}_m) - r_f}{\sigma_m}$, which can be solved for $E(\tilde{r}_i)$ to give us the standard Sharpe-Lintner-Mossin CAPM[1]:

$$E(\tilde{r}_i) = r_f + \beta_i\left[E(\tilde{r}_m) - r_f\right], \text{ where } \beta_i = \frac{\sigma_{i,m}}{\sigma_m^2}$$

[1] The CAPM was first published by a Stanford economist/mathematician, William Sharpe (and he shared the 1990 Nobel Prize in Economics with Harry Markowitz, whose work we explored in the last lecture). However, independent of Sharpe, two other researchers, John Lintner, and Jan Mossin, also came up with the same model. Hence, the CAPM is also referred to as the Sharpe-Lintner-Mossin CAPM.

An intuitive proof:

In the last lecture, we showed that the tangency or MVE portfolio has a nice property. Let us recall that property:

$$\frac{E(\tilde{r}_X)-r_f}{Cov(\tilde{r}_X,\tilde{r}_{MVE})}=\frac{E(\tilde{r}_Y)-r_f}{Cov(\tilde{r}_Y,\tilde{r}_{MVE})}=\frac{E(\tilde{r}_Z)-r_f}{Cov(\tilde{r}_Z,\tilde{r}_{MVE})}$$

Since we now know that the MVE is the market, we can write;

$$\frac{E(\tilde{r}_X)-r_f}{Cov(\tilde{r}_X,\tilde{r}_m)}=\frac{E(\tilde{r}_Y)-r_f}{Cov(\tilde{r}_Y,\tilde{r}_m)}=\frac{E(\tilde{r}_Z)-r_f}{Cov(\tilde{r}_Z,\tilde{r}_m)}$$

This property says that the ratio of the risk premium of any asset to its covariance with the tangency portfolio is constant. Note that this is a general property that holds for any asset and any portfolio of assets. In particular, let us write this formula for any asset *i* and the market portfolio *m* (which is a perfectly valid portfolio).

$$\frac{E(\tilde{r}_i)-r_f}{Cov(\tilde{r}_i,\tilde{r}_m)}=\frac{E(\tilde{r}_m)-r_f}{Cov(\tilde{r}_m,\tilde{r}_m)}$$

But we know (see statistical review in the last lecture) that $Cov(\tilde{r}_m,\tilde{r}_m)=Var(\tilde{r}_m)=\sigma_m^2$. This implies that: $\dfrac{E(\tilde{r}_i)-r_f}{\sigma_{i,m}}=\dfrac{E(\tilde{r}_m)-r_f}{\sigma_m^2}$, which leads to: $E(\tilde{r}_i)=r_f+\beta_i\left[E(\tilde{r}_m)-r_f\right]$, the standard CAPM.

- **What the CAPM is saying in plain English**

1) Covariance (with market) , not variance is the appropriate measure of risk.

   Why? Because the covariance of an asset with a portfolio can be interpreted as the *marginal variance*. To see this go back to the proof of the CAPM. There, we saw that when we add $\$\alpha$ worth of asset *i* to the market, the standard deviation of this portfolio is given by equation (B):

$$\sigma_p=\left[\sigma_m^2+\alpha^2.\sigma_i^2+2\alpha\sigma_{i,m}\right]^{1/2} \qquad \ldots \qquad \text{(B)}$$

Along the way we took the derivative of this w.r.t. α. This derivative gives us the marginal increase in portfolio variance by adding \$$\alpha$ worth of asset *i*. This marginal increase in variance is given by equation (D):

$$\frac{d\sigma_p}{d\alpha} = \frac{1}{\sigma_p}\left[\alpha.\sigma_i^2 + \sigma_{i,m}\right] \qquad \ldots \qquad \text{(D)}$$

When we evaluate this derivative at α=0, i.e. adding very small amounts of asset i, we obtain: $\left.\frac{d\sigma_p}{d\alpha}\right|_{\alpha=0} = \frac{\sigma_{i,m}}{\sigma_m}$. Notice that this term does not depend at all on σ_i, the standard deviation of the asset! *The marginal contribution of the asset to the portfolio's variance is entirely dependent upon the covariance of the asset i with the market portfolio,* $\sigma_{i.m}$[2]. *Beta* (covariance normalized by the variance of market return) is thus the relevant measure of systematic risk.

2) <u>High beta assets have high expected returns</u>

   According to the CAPM, there is a linear relationship between beta and expected return. What is the economic intuition behind this result? High beta assets have high returns when the overall market return is high i.e. they pay off well when you least need the money. Conversely, when you need the money most (which is when the overall market is doing badly), these assets fare poorly. So, in equilibrium, these assets have to offer high expected returns in order that investors hold them.

3) <u>Beta alone determines expected returns</u>

   Once beta is taken into account, no other factors determine expected returns. Beta is the sole factor affecting security prices and returns. As we

[2] *Marginal*, and not average or total quantities are of great significance in economic reasoning. e.g. In the early hours of a final exam day, after the tenth cup of coffee, what every grad student evaluates is the *marginal benefit* of one more hour's study (in terms of incremental points) versus the *marginal cost* (decreased effectiveness due to lost sleep leading to bad performance) of that hour's study.

shall see, this is the implication that has proved to be largely incompatible with *historical* data on stock returns.

4) Diversifiable risk is not priced

   The best way to see this is to consider an asset that has a high variance, but a zero covariance with the market portfolio. The CAPM says that this asset, in spite of its high volatility, will have an expected return equal to the risk-free rate, as its beta is zero. This means all the risk of this stock is idiosyncratic, which means it can be diversified away. Since diversifying is costless (e.g. one can buy into a no-load *mutual fund*), investors are not compensated for such idiosyncratic or company-specific risk. All that is priced is *systematic risk*, measured by beta.

- **The Security Market Line**
  - The Security Market Line (hereafter, SML) is a graphical representation of the CAPM pricing relation. It is best to view the SML for an example problem. For this, let's recall the three-asset example from the last lecture.
  - You will recall that we plotted the efficient frontier for assets X, Y and Z, and found the tangency portfolio to have the following proportions of the three assets: $w_{X,MVE} = 22.74\%$, $w_{Y,MVE} = 177.93\%$, $w_{Z,MVE} = -100.67\%$
  - From this lecture, we know that in a world with just these three risky assets, this MVE portfolio must be the market portfolio. However, the above MVE portfolio is not a plausible market portfolio. (Why?)
  - Even though the math is correct, the resulting portfolio does not have sensible properties of a market portfolio. So, I tweaked with the correlations a bit, and worked with the following assumptions:

$$\mu = \begin{bmatrix} 0.10 \\ 0.20 \\ 0.15 \end{bmatrix}, \text{ and } \Sigma = \begin{pmatrix} 0.0049 & 0.0032 & 0.0013 \\ 0.0032 & 0.0100 & 0.0054 \\ 0.0013 & 0.0054 & 0.0144 \end{pmatrix}$$

I solved the Markowitz problem, and came up with the following MVE, or market portfolio: $w_{X,m} = 5.46\%,\ w_{Y,m} = 84.24\%,\ w_{Z,m} = 10.30\%$. *This* looks like a plausible market portfolio.

- Now, we proceed to calculate the betas of each of the assets with respect to the market portfolio we just obtained. Calculations are shown below:

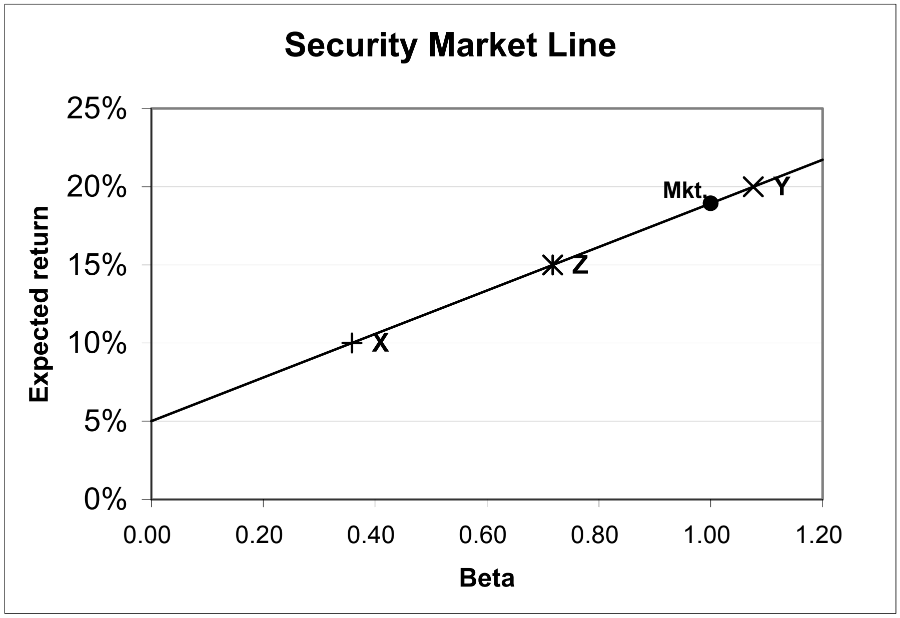
Beta calculations					**Risk**	**Tangency**
Asset *i*	**Cov(r_i,r_m)**	**Var(r_m)**	**Beta (β_i)**	***E*(r_i)**	**Premium**	**Ratio**
r_f	0.0000	0.0085	0.00	5.00%	0.00%	-
Mkt	0.0085	0.0085	1.00	18.94%	13.94%	16.3895
X	0.0031	0.0085	0.36	10.00%	5.00%	16.3895
Y	0.0092	0.0085	1.08	20.00%	15.00%	16.3895
Z	0.0061	0.0085	0.72	15.00%	10.00%	16.3895

- It can be seen from the above table that beta and expected return go hand in hand.

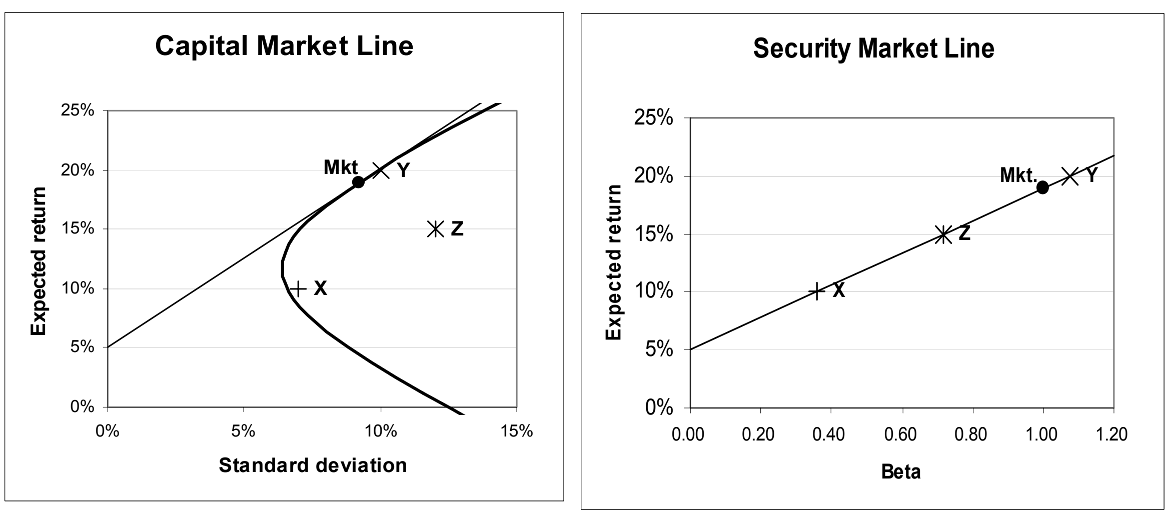
- As a useful aside, I have also calculated the tangency ratio $\dfrac{E(\tilde{r}_i) - r_f}{Cov(\tilde{r}_i, \tilde{r}_{MVE})}$ using the last two columns of the above table, , and we can see that:

$$\frac{E(\tilde{r}_X) - r_f}{Cov(\tilde{r}_X, \tilde{r}_{MVE})} = \frac{E(\tilde{r}_Y) - r_f}{Cov(\tilde{r}_Y, \tilde{r}_{MVE})} = \frac{E(\tilde{r}_Z) - r_f}{Cov(\tilde{r}_Z, \tilde{r}_{MVE})} = \frac{E(\tilde{r}_{MVE}) - r_f}{Cov(\tilde{r}_{MVE}, \tilde{r}_{MVE})}$$

- As usual, a picture is a better way to understand these numbers. So I graphed the SML with beta on the horizontal axis, and expected return on the vertical axis.



- This line contains the essence of the CAPM. Under the equilibrium conditions assumed by the CAPM, *all assets* should lie on the security market line. The higher the beta, the higher the expected return.
- It is instructive to put the CML and the SML beside each other and compare them. This is what I have done in the following picture.



- Compare the two panels of the figure. The vertical axis on both panels represents the mean, or expected return $E(\tilde{r})$. The only difference in these diagrams is the horizontal axis. In the left panel this axis represents the standard deviation, which means this panel is a mean-standard deviation diagram. In the right panel, the horizontal axis represents the beta, or systematic risk. This is a mean-beta diagram.

- All assets lie on the SML. Except the market portfolio and the risk-free asset, no other asset lies on the CML.
- The CML plot shows total risk (systematic and unsystematic), while points on the SML show only systematic risk.
- While investments with the same mean can have different standard deviations, they must have the same beta. Since mean return and beta plot on a straight line, all investments with the same mean return must have the same beta, and vice versa. For instance, in Panel A, all points on the horizontal line to the right of the *market portfolio*, have the same beta as the market, which is 1. However, of all such points (or portfolios), the market portfolio has the minimum standard deviation. Thus the entire line of points in the mean-standard deviation diagram plots at one point on the mean-beta diagram.

- **Beta as a regression coefficient**
  - Consider the regression equation:

$$\tilde{r}_i - r_f = \alpha_i + \beta_i(\tilde{r}_m - r_f) + \varepsilon_i \qquad \ldots(1)$$

  - That is we are "regressing" the excess return on asset i on the excess return of the market portfolio. At this point this is just an arbitrary equation relating the excess returns on asset i and that on the market. We shall see that the CAPM implies some special things about the ε_i.
  - Taking expectations of (1), we get:

$$E(\tilde{r}_i) - r_f = \alpha_i + \beta_i(E(\tilde{r}_m) - r_f) + E(\varepsilon_i) \qquad \ldots(2)$$

  Comparing (2) to the CAPM: $E(\tilde{r}_i) - r_f = \beta_i(E(\tilde{r}_m) - r_f)$, we can deduce that:

$$\alpha_i = 0; E(\varepsilon_i) = 0 \qquad \ldots (3)$$

So we can rewrite (1) as $\tilde{r}_i = r_f + \beta_i(\tilde{r}_m - r_f) + \varepsilon_i$ … (1′)

- Next, we compute the covariance between $\tilde{r}_i$ and $\tilde{r}_m$:

$$Cov(\tilde{r}_i, \tilde{r}_m) = Cov(r_f + \beta_i(\tilde{r}_m - r_f) + \varepsilon_i, \tilde{r}_m) = \beta_i.\sigma_m^2 + Cov(\varepsilon_i, \tilde{r}_m)$$

But from the definition of beta, we have $Cov(\tilde{r}_i, \tilde{r}_m) = \beta_i.\sigma_m^2$.

This must imply that $Cov(\varepsilon_i, \tilde{r}_m) = 0$ … (4)

- Computing the variance of asset i's return, using (1′), we can write:

$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_\varepsilon^2 + \beta_i Cov(\varepsilon_i, \tilde{r}_m) = \beta_i^2\sigma_m^2 + \sigma_\varepsilon^2 \quad \ldots (5)$$

- What is the meaning of equations (3), (4) and (5)?

1) Let us start with (5) first. It means that the variance of an asset's return is the sum of two parts. The first part $\beta_i^2\sigma_m^2$ is called the *systematic risk* of asset *i*. It is the risk of the market as a whole, and cannot be diversified away. The second part σ_ε^2 is called the unsystematic, or *idiosyncratic risk* of asset i. This is the part of the total risk of the asset that we can get rid of by diversification.
2) Equation (4) means that the nature of idiosyncratic risk is such that it is uncorrelated with the market. Hence, when one aggregates all risky assets in the market portfolio, all such unsystematic risks cancel out, and we are left with a well-diversified portfolio that has only systematic risk.
3) Equation (1′) says that the (random) risk premium on a stock consists of a systematic term $\beta_i(\tilde{r}_m - r_f)$, and a purely random term ε_i. From equation (3) we learn that the expected, or average value of ε_i is zero i.e. we will have our asset outperform the CAPM expected return in some periods, and underperform it in some periods. Over a long period of time, one cannot hope to earn more than the expected return that the CAPM says it should earn.

- **Notes on beta**
  - The definition of beta is: $\beta_i = \dfrac{Cov(\tilde{r}_i, \tilde{r}_m)}{\sigma_m^2}$. This means that the beta is really a measure of the covariance of an asset's return with the return on the market. It can also be understood (from the regression discussion above) as the sensitivity of the asset's return to the return on the market.
  - One should always remember that:
    1) β of risk-free asset = β_{rf} = $Cov(r_f, r_m)/Var(r_m)=0$, as the risk-free asset has a covariance of zero with any other asset, including the market portfolio.
    2) β of the market portfolio= β_m= $Cov(r_m, r_m)/Var(r_m)$= $Var(r_m)/Var(r_m)=1$, as the covariance of any asset with itself is the variance of that asset.
  - A nice property of beta is *linearity*: the *beta* of a portfolio is the weighted average of the betas of the components of that portfolio. i.e.

$$\text{Portfolio beta } = \beta_p = \sum_{i=1}^{n} w_i.\beta_i = (w_1.\beta_1 + w_2.\beta_2 + \cdots + w_n.\beta_n) .$$

- **The certainty equivalent form of the CAPM**
  - The CAPM is an asset *pricing* model, but the pricing model is expressed in terms of expected returns. Can we derive a form of the CAPM that deals with prices rather than returns?
  - Suppose that the random future price of an asset is $\tilde{F}$, and its current price *P*. Then, the expected return on this asset is $E(\tilde{r}) = \dfrac{E(\tilde{F}) - P}{P} = \dfrac{E(\tilde{F})}{P} - 1$.

    The CAPM says that $E(\tilde{r}) = r_f + \beta(E(\tilde{r}_m) - r_f)$. Equating these two definitions of expected return, we get:

$$\frac{E(\tilde{F})}{P} = 1 + r_f + \beta(E(\tilde{r}_m) - r_f) \qquad \ldots \qquad (6)$$

Beta is defined as: $\beta = \dfrac{Cov(\tilde{r},\tilde{r}_m)}{\sigma_m^2} = \dfrac{Cov(\frac{\tilde{F}}{P}-1,\tilde{r}_m)}{\sigma_m^2} = \dfrac{Cov(\tilde{F},\tilde{r}_m)}{P\sigma_m^2}$ … (7)

Now, combining (6) and (7), we get:

$$\frac{E(\tilde{F})}{P} = 1 + r_f + \frac{Cov(\tilde{F},\tilde{r}_m)}{P\sigma_m^2}(E(\tilde{r}_m) - r_f),$$

which simplifies to:

$$P = \frac{1}{1+r_f}\left[E(\tilde{F}) - \frac{Cov(\tilde{F},\tilde{r}_m)(E(\tilde{r}_m)-r_f)}{\sigma_m^2}\right]$$

- This is called the certainty equivalent (CE) form of the CAPM. The important thing about this formula is that it is linear in $\tilde{F}$. To see this assume two assets with uncertain future values $\tilde{F}_1$ and $\tilde{F}_2$. Using this CE form of the CAPM, we can show that $P(\tilde{F}_1 + \tilde{F}_2) = P(\tilde{F}_1) + P(\tilde{F}_2)$ i.e. the price of a package of assets is equal to the sum of the prices of the individual assets (i.e. the price of a happy meal is the sum of the individual prices of the burger, fries and the Coke).
- Linearity in pricing is crucial as it precludes the existence of *arbitrage* opportunities. Why? If the happy meal were selling for a lower price than the sum of the prices of the components. We could buy the happy meal, and sell the burger, fries and the Coke separately (at their individual prices), and make a riskless profit. This would be an *arbitrage* opportunity.
- It is a fundamental tenet of financial theory that *arbitrage opportunities can be precluded if, and only if, the pricing is linear* (This statement can be proved rigorously). This is an important principle, illustrated in this case by the CAPM.