FINC 748: Investments	**Ramana Sonti**
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Lecture Note 3B: Optimal risky portfolios

To be read with BKM Chapter 8

- Statistical Review
- Portfolio mathematics
- Mean standard deviation diagrams
- Mean variance analysis: 2 risky assets
- Mean variance analysis: 3 risky assets
- Overall portfolio optimization

- **Statistical Review**
  - Before we jump head first into risky portfolios, it is useful to review some basic statistics. We shall focus on the following quantities.
    - ✓ Expected Return
    - ✓ Variance and Standard Deviation
    - ✓ Covariance and Correlation
  - When an asset (a stock, a bond etc.) is originally acquired, its rate of return is usually uncertain, that is it can take any value. So, we consider return $\tilde{r}$ to be a random variable (henceforth, RV). As before, the tilde on top indicates the "random variable" nature of return.
  - If a RV can take on any one of a finite number of specific values, it is called *discrete*. Associated with each possible value that the RV can take is a *probability*. The simplest examples of discrete RVs are coin tosses or rolls of a die. In the former example, *possible outcomes* are "Head" or "Tail", each possible with a probability of 0.5. In the die example, possible outcomes are 1,2,….6, each with a probability of 1/6.
  - If a RV can take any real value in an interval, as for example the temperature in a room, then it is called a *continuous* RV. Here you cannot make a list of all possible outcomes. We can calculate the probability of the RV's value lying *within any range* using a *probability density function*. A popular continuous random variable is the Normal random variable.
  - Asset returns are obviously continuous, as they can take on an infinite number of values. However, to simplify exposition, we will talk of returns *as if* they were discretely distributed. That is to say, we *assume* that the future can be divided into a finite number of *states*, and *assume* that the asset's return will be a certain number in a given state. We also *assume* a probability for each state.

<u>Example</u>: Consider the following two assets

Returns State	Prob.	Tech Stock	Gold
Boom	1/3	0.60	-0.70
Normal	1/3	0.20	0.05
Recession	1/3	-0.50	0.50

Here we conveniently split the future into three possible *states* – "boom", "normal" and "recession". Now, we can proceed to understand some statistics.

✓ Expected (or Mean) Return, $E(\tilde{r}) = \sum_{i=1}^{S} p_i.\tilde{r}_i$, where $\tilde{r}_i$ is the (possible) return in each state, and p_i is the probability of that state.

The expected return is simply as probability-weighted average of the asset's return in each state. We use expected return as a measure of the average return expected from the asset.

The expected return of the tech. stock is:

$E(\tilde{r}_{ts})$ = (1/3 x 0.60)+(1/3 x 0.20)+(1/3 x –0.50) = 0.10=10%

The expected return of Gold is:

$E(\tilde{r}_g)$ = (1/3 x -0.70)+(1/3 x 0.05)+(1/3 x 0.50) = -0.05 = -5%

✓ Variance, $\sigma^2(\tilde{r}) = E\left[(\tilde{r}_i - E(\tilde{r}))^2\right] = \sum_{i=1}^{S} p_i.(\tilde{r}_i - E(\tilde{r}))^2$

✓ Standard deviation, $\sigma(\tilde{r}) = \sqrt{Variance}$

The variance is a probability weighted measure of (squared) deviation of the asset return from the mean return in each state. Standard deviation is easier to interpret as it has the same units as the asset return. We use variance and standard deviation to measure the *risk* of an asset's return.

Calculation of the variance of the tech. stock is as follows:

TECH. STOCK				**Squared**	
State	**Prob.**	**Return**	**Deviation**	**Deviation**	**Product**
Boom	1/3	0.60	0.50	0.2500	0.0833
Normal	1/3	0.20	0.10	0.0100	0.0033
Recession	1/3	-0.50	-0.60	0.3600	0.1200
				Variance=	**0.2067**
				Std. Deviation=	**45.46%**

Calculation of the variance of the tech. stock is as follows:

GOLD				**Squared**	
State	**Prob.**	**Return**	**Deviation**	**Deviation**	**Product**
Boom	1/3	-0.70	-0.65	0.4225	0.1408
Normal	1/3	0.05	0.10	0.0100	0.0033
Recession	1/3	0.50	0.55	0.3025	0.1008
				Variance=	**0.2450**
				Std. Deviation=	**49.50%**

Or, one could use the equation directly to get the variances and standard deviations.

- **Portfolio Math: A first look at diversification**
  - We can see from the numbers that Gold has a lower return than the tech stock, *and* has a higher variance (and standard deviation) than that of the tech stock. Obviously, on its own, it is a worse investment than the tech stock.
  - Let us see what happens when we form a *portfolio* (combination) of the tech. stock and gold in the ratio 75%:25%. In particular, let's look at the expected return and the variance of the portfolio.

Portfolio Calculations: Method 1

- In this method, we form the portfolio first and then find the expected return and variance of the portfolio, like any other asset.
- The portfolio returns in each state can be found as:

  Boom: $\tilde{r}_p$ = (0.75 x 0.60) + (0.25 x –0.70) = 0.2750

  Normal: $\tilde{r}_p$ = (0.75 x 0.20) + (0.25 x 0.05) = 0.1625

  Recession: $\tilde{r}_p$ = (0.75 x -0.50) + (0.25 x 0.50) = -0.2500

- Now, we can easily calculate the Expected Return and Variance (and Standard Deviation) of this portfolio (which is like any other asset), as follows:

Portfolio of Tech. Stock and Gold: Calculations for Method I					
State	**Prob.**	**Return**	**Deviation**	**Squared Deviation**	**Product**
Boom	1/3	0.2750	0.2125	0.0452	0.0151
Normal	1/3	0.1625	0.1000	0.0100	0.0033
Recession	1/3	-0.2500	-0.3125	0.0977	0.0326
	Mean=	**6.25%**		**Variance=**	**0.0509**
				Std. Deviation=	**22.57%**

- Points to notice:
  1. The variance of the portfolio is *much lower* than the variance of either asset by itself. [Compare 0.0509 for the σ^2(portfolio) to 0.2067 for σ^2(stock) and 0.2450 for σ^2(gold)]. This is the effect of *diversification* (putting your money in more than one asset).
  2. The intuitive reason for this is that the stock and the gold move in opposite directions to each other and cancel each other's fluctuations out. The technical term for this is that the *covariance between these assets is negative.* This is the reason why diversification works.
  3. As we keep adding more assets into portfolio, diversification drastically reduces the variance and standard deviation of the portfolio – up to a point, after which no further reduction is possible.

Portfolio Calculations: Method 2

- In this method, we use formulas to directly obtain the Expected Return and Variance of the portfolio.
- The formulas, for any portfolio consisting of two assets A and B, are:
  - ✓ Expected Return: $E(\tilde{r}_p) = w_A.E(\tilde{r}_A) + w_B.E(\tilde{r}_B)$
  - ✓ Variance: $$\begin{aligned}\sigma^2(\tilde{r}_p) &= w_A^2.\ \sigma^2(\tilde{r}_A) + w_B^2.\ \sigma^2(\tilde{r}_B) + 2.w_A.w_B.Cov(\tilde{r}_A, \tilde{r}_B) \\ &= w_A^2.\ \sigma^2(\tilde{r}_A) + w_B^2.\ \sigma^2(\tilde{r}_B) + 2.w_A.w_B.\rho_{A,B}\sigma(\tilde{r}_A)\sigma(\tilde{r}_B)\end{aligned}$$

  Note 1: Here, w_A and w_B are the weights (or proportions) of the 2 assets in the portfolio. Remember that $w_A+w_B=1$, always.

  Note 2: $Cov(\tilde{r}_A, \tilde{r}_B)$ is called the *covariance* between asset returns $\tilde{r}_A$ and $\tilde{r}_B$ and measures how, and by how much, these returns move together. In line 2 of the variance formula above, we expand the formula of covariance: $Cov(r_A,r_B)= \rho_{A,B}.\sigma(r_A).\sigma(r_B)$, where $\rho_{A,B}$ is known as the correlation between asset returns r_A and r_B.

  Note 3: In fact, equations (1) and (2) of Lecture Note 3A are special cases of these two equations.

- In our example above, if we let asset A be the Stock and asset B be Gold, then:

  Expected Return, $E(r_p) = w_A.E(r_A) + w_B.E(r_B) = 0.75 \times 10\% + 0.25 \times -5\% = 6.25\%$, which agrees with the calculation by Method 1.

  To use the portfolio variance formula, we need to calculate covariance first: Covariance between r_A and r_B is given by:

$$Cov(\tilde{r}_A, \tilde{r}_B) = \sum_{i=1}^{S} p_i.\left(\tilde{r}_{A,i} - E(\tilde{r}_A)\right)\left(\tilde{r}_{B,i} - E(\tilde{r}_B)\right)$$

  The following table gives the covariance calculation for our two assets:

Covariance and correlation between tech. Stock and Gold					
State	**Prob.**	**Devn. of Tech.**	**Devn. of Gold**	**Product of Devns.**	**Product**
Boom	1/3	0.50	-0.65	-0.3250	-0.1083
Normal	1/3	0.10	0.10	0.0100	0.0033
Recession	1/3	-0.60	0.55	-0.3300	-0.1100
				Covariance=	**-0.2150**
				Correlation=	**-0.96**

Note 1: The sign of the correlation (or the covariance) tells us which way the two returns are moving *relative to each other*. A positive number means the assets move *together*, and a negative number means they move *opposite* to each other.

Note 2: Variance can be viewed as a special case of covariance, i.e.

$$Cov(\tilde{r}_i, \tilde{r}_i) = Var(\tilde{r}_i) = \sigma^2(\tilde{r}_i)$$

Note 3: Covariance depends upon the units of return, while correlation does not

Note 4: Correlations, by definition, have to be greater than –1 and lesser than +1, i.e. $-1 \leq \rho \leq +1$.

Thus, the stock and the gold in our example are *almost perfectly negatively correlated*. That's good news for diversification!

- Now, we can calculate the variance of the portfolio using the formula above:

$$\sigma^2(r_p) = (0.75)^2.(0.2067) + (0.25)^2.(0.2450) + 2.(0.75)(0.25).(-0.2150)$$

= 0.0509, which agrees with the calculation by Method 1.

- In general, for N assets, the formulas are:

$$E(\tilde{r}_p) = \sum_{i=1}^{N} w_i . E(\tilde{r}_i) \text{, and}$$

$$\sigma^2(\tilde{r}_p) = \sum_{i=1}^{N} w_i^2 . \sigma^2(\tilde{r}_i) + 2. \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} w_i . w_j . Cov(\tilde{r}_i . \tilde{r}_j) \text{, with } \sum_{i=1}^{N} w_i = 1$$

- **Two Risky Assets; No risk-free asset**

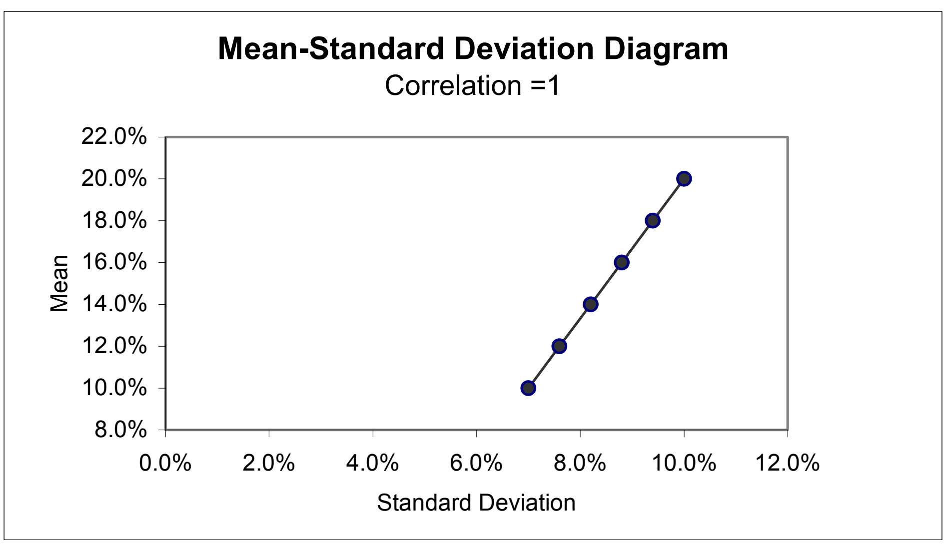
Let us kick off our analysis with portfolios of two risky assets, in the absence of any risk-free assets. Consider assets X and Y. X has an *E(r)* of 10% and a variance of 0.0049, which implies a standard deviation of 0.07 or 7%. Asset Y has an *E(r)* of 20% and a variance of 0.0100, which implies a standard deviation of 0.10 or 10%. Let us vary the weights (proportions of X and Y) of this portfolio, and observe what happens to the $E(r_p)$ and $\sigma(r_p)$ of the portfolio

Case 1: Correlation ($\rho_{X,Y}$) = 1

$$E(r_p) = w_X(0.10)+w_Y(0.20), \text{ and}$$

$$\sigma^2(r_p) = w_X^2 (0.0049)+w_Y^2(0.0100)+2w_Xw_Y(1)(0.07)(0.10)$$

Each value of w_X (and hence w_Y), gives us one point in the *mean-standard deviation space*.

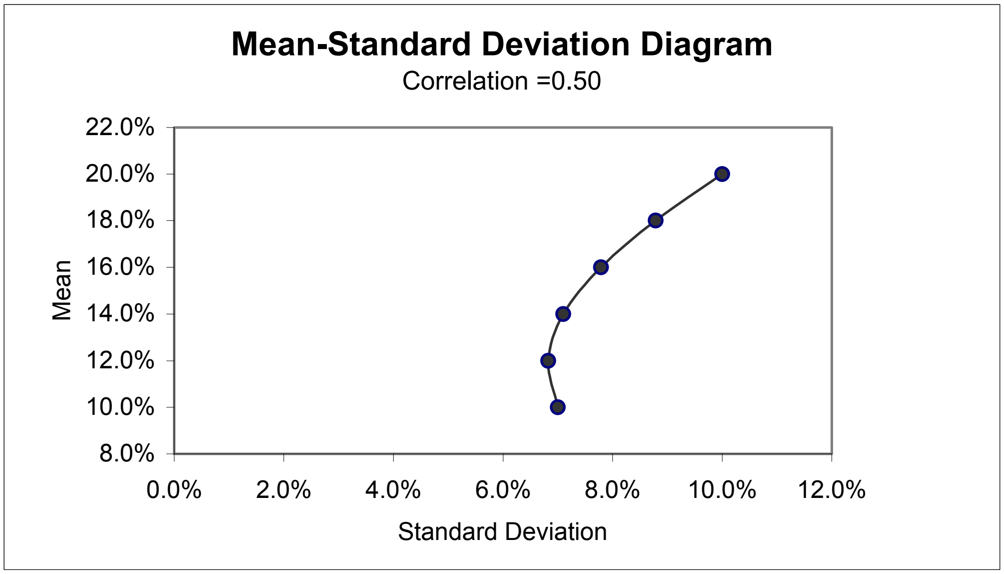


<u>Case 2</u>: Correlation ($\rho_{X,Y}$) = 0.5

$$E(r_p) = w_X(0.10)+w_Y(0.20), \text{ and}$$

$$\sigma^2(r_p) = w_X^2\,(0.0049)+w_Y^2(0.0100)+2w_Xw_Y(0.5)(0.07)(0.10)$$

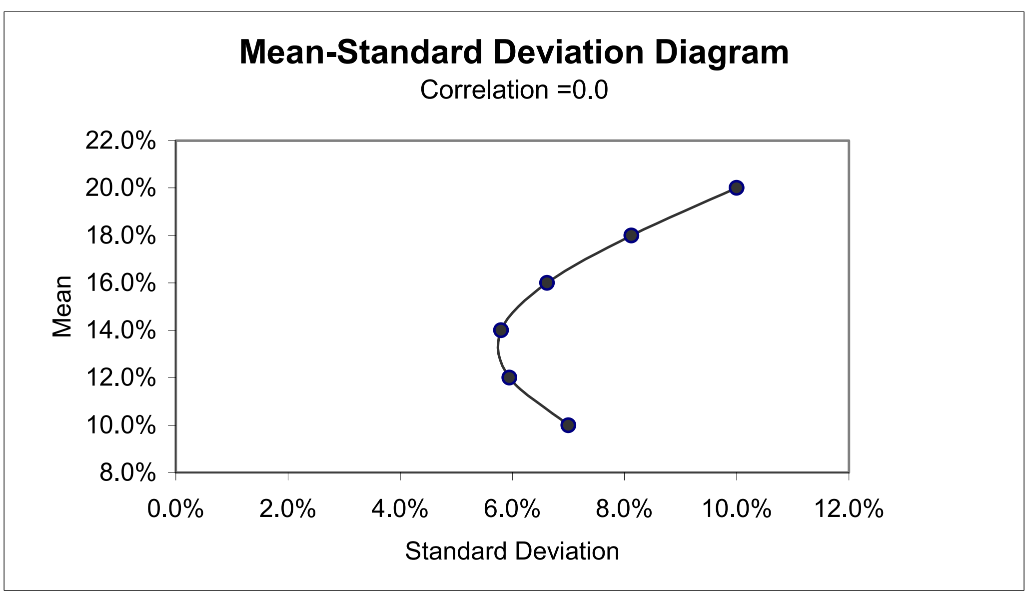
Once again, each value of w_X (and hence w_Y), gives us one point in the *mean-standard deviation space*.



<u>Case 3</u>: Correlation ($\rho_{X,Y}$) = 0.0

$$E(r_p) = w_X(0.10)+w_Y(0.20), \text{ and}$$

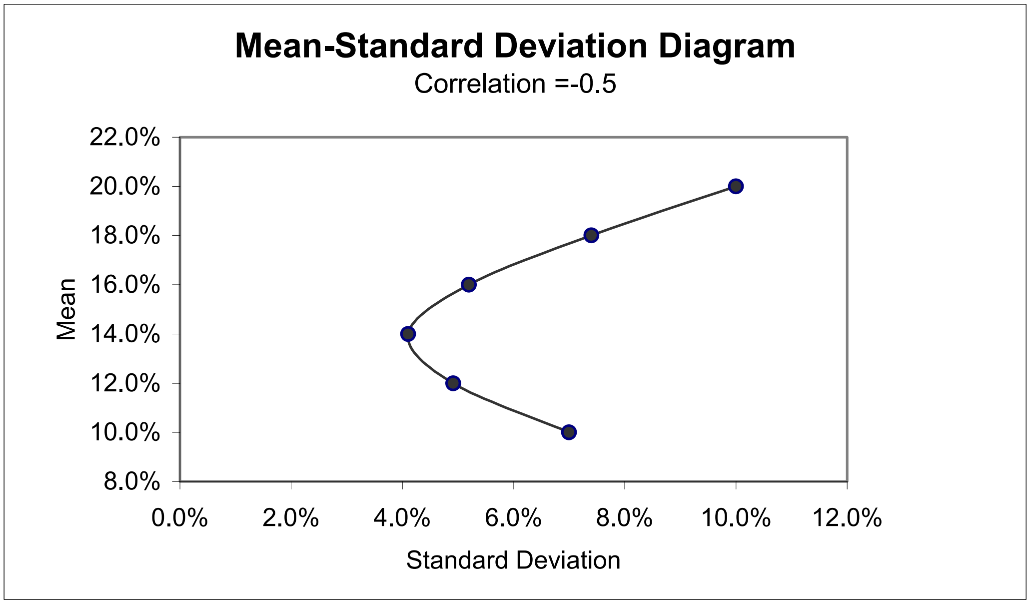
$$\sigma^2(r_p) = w_X^2\,(0.0049)+w_Y^2(0.0100)+2w_Xw_Y(0)(0.07)(0.10)$$



<u>Case 4</u>: Correlation ($\rho_{X,Y}$) = -0.5

$E(r_p) = w_X(0.10)+w_Y(0.20)$, and

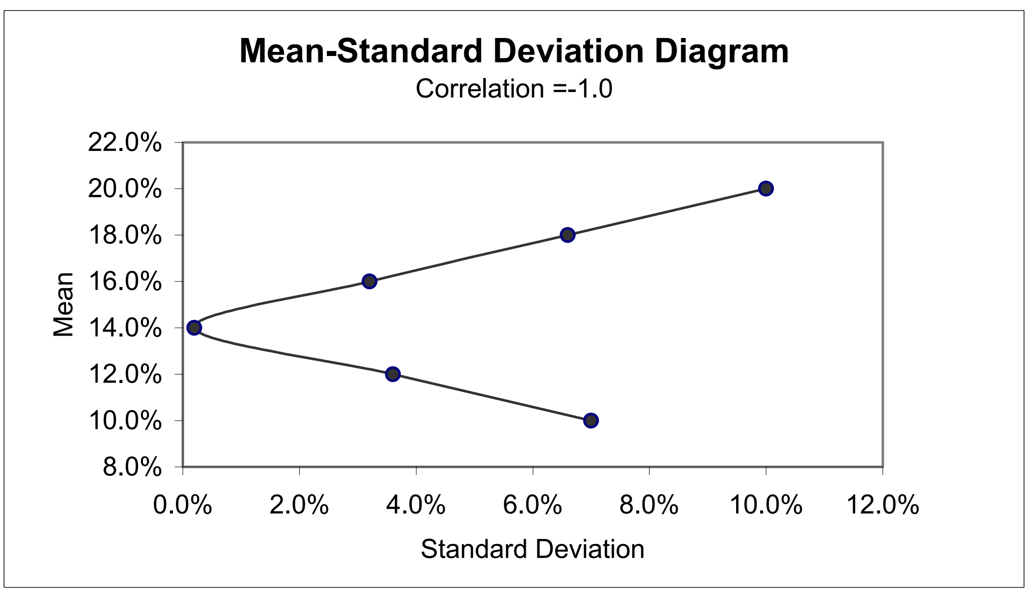
$\sigma^2(r_p)=w_X^2(0.0049)+w_Y^2(0.0100)+2w_Xw_Y(-0.5)(0.07)(0.10)$



<u>Case 5</u>: Correlation ($\rho_{X,Y}$) = -1.0

$E(r_p) = w_X(0.10)+w_Y(0.20)$, and

$\sigma^2(r_p) = w_X^2\ (0.0049)+w_Y^2(0.0100)+2w_Xw_Y(-1)(0.07)(0.10)$



<u>Points to be noted from this exercise</u>:

1. The end-points of the diagram are the two assets X and Y themselves, which is simply saying that a portfolio of 100% X and 0% Y is just the asset X. Conversely, a portfolio of 0% X and 100% Y is just the asset Y. So, whatever the correlation, the end-points remain rooted to their spot (like the two pegs of a clothesline).
2. When correlation is +1.0, the mean-standard deviation diagram is simply a straight line. For those mathematically oriented, when correlation =1.0, we have: $\sigma^2(r_p) = w_X^2\,(0.0049)+w_Y^2(0.0100)+2w_Xw_Y(1)(0.07)(0.10)$

   $= (0.07w_X)^2 + (0.1w_Y)^2+2(0.07w_X)(0.10w_Y)$

   $= (0.07w_X+0.1w_Y)^2$, by the $(a+b)^2=a^2+b^2+2ab$ formula

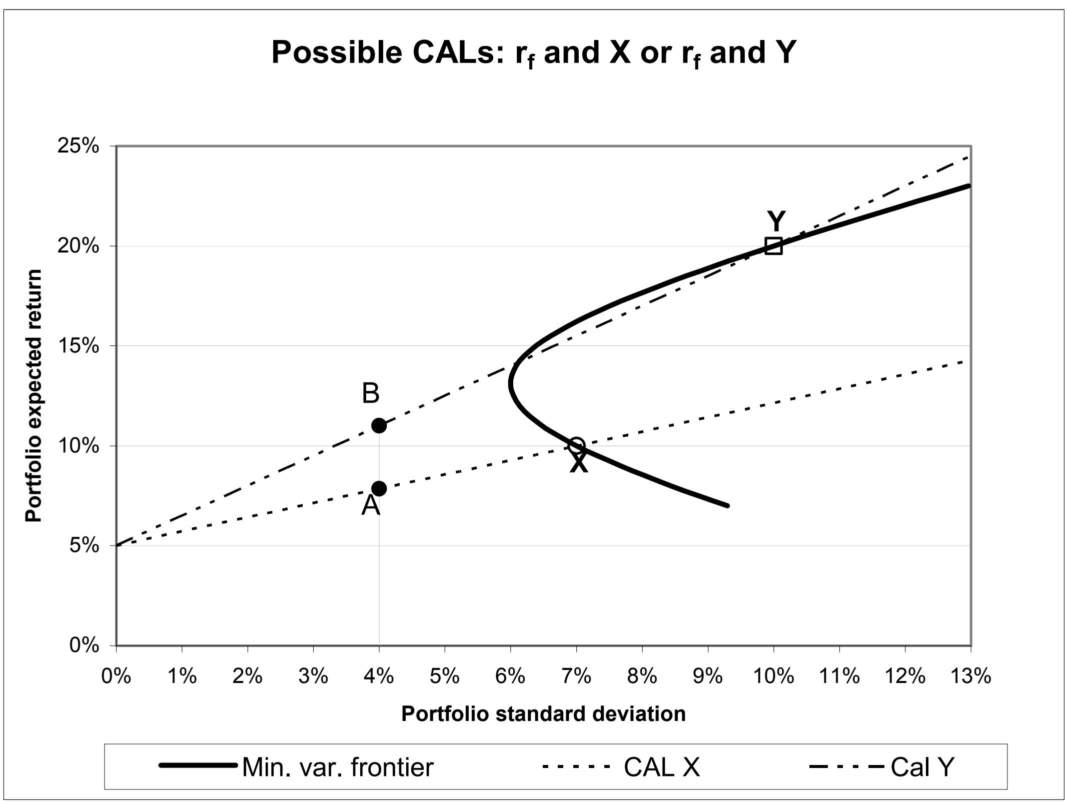
   $\Rightarrow \sigma(r_p) = (0.07w_X+0.1w_Y)$, which is a linear combination of the standard deviations of the two assets. For a correlation of –1.0, a similar logic results in $\sigma(r_p) = (0.07w_X - 0.1w_Y)$ (try this!), which is also linear. Hence we have two straight lines meeting on the Y-axis (return axis), which we see in Case 5.

   For correlations other than –1.0 and +1.0, the portfolio standard deviation is not a linear function of the standard deviations of the two assets, and we don't have straight lines, but curves.
3. As we keep decreasing the correlation from +1.0 toward –1.0, the diagram curves in towards the left (the end-points are still fixed). This is because, as we decrease the correlation, we have *diversification*. i.e. we have some combinations of these two assets which have the lower standard deviations than the two assets by themselves, for each given return.
4. As we change the correlations, the standard deviation is the only thing that changes. The portfolio expected return (mean return) does not change. Why? Because it is $E(r_p) = w_X(0.10)+w_Y(0.20)$, which does not depend upon correlation at all. So, in every curve, the height of the 6 dots representing each portfolio remains the same.

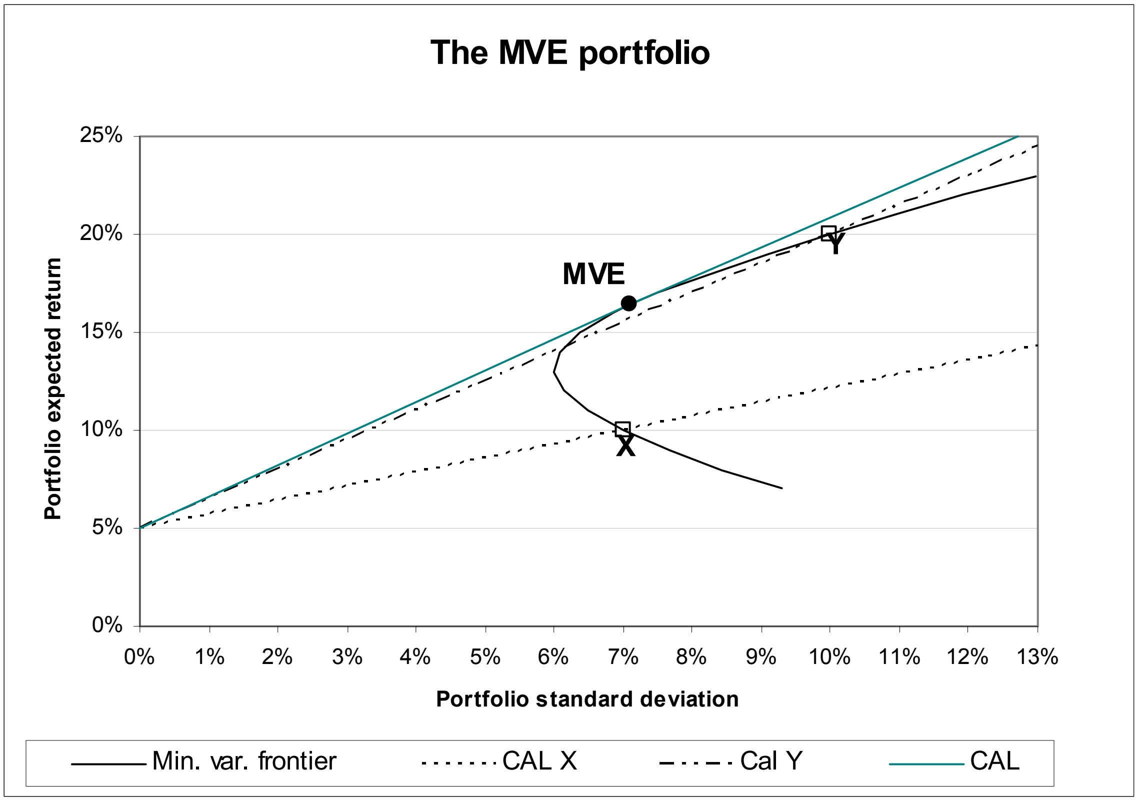
- So, with sufficiently low correlations, it is possible that portfolios formed from our two assets have lower variance (and standard deviation) than either asset by itself.
- Now, there are many, many portfolios that can be formed from these two assets. Varying the weights on the two assets (while making sure the sum of both the weights is 1), gives us the full *investment opportunity set with only risky assets.* This curve is also known as the *mean-variance frontier*, for reasons that will soon become evident.

▪ **Two Risky Assets; One risk-free asset**

- Now, let's throw our old risk-free asset with r_f = 5% into the mix. Also, let's fix the correlation between the returns of assets X and Y at ρ = 0.10. The following diagram plots two possible *Capital Allocation Lines*.



- Let's start with CAL Y. We can see that by adding the risk-free asset to Asset Y, we can do better than by doing the same with Asset X. For every level of standard deviation (risk) on CAL X, there is a corresponding point on CAL Y (vertically above it) that has a higher expected return. For example, at a standard deviation level of 4%, one could be at portfolio A on CAL X with an expected return of close to 8%; however, at the same level of risk, one could be at portfolio B on CAL Y with an expected return of 11%. Obviously, a rational investor would prefer to be at B on CAL Y.
- Since the same logic holds for every point of CAL X and every corresponding point on CAL Y, we say that *CAL Y dominates CAL X from a mean-variance standpoint*.
- Now, we are clear that pairing the risk-free asset with Asset Y is much better than pairing it with Asset X. But, what if we are willing to consider all possible *portfolios* of X and Y (not just X or Y) to pair with the risk-free asset? Which would be the *optimal risky portfolio* then?
- To find out, imagine starting with CAL X and pivoting it counter-clockwise about the risk-free asset. Soon you will come to CAL Y. If you continue further, is there a CAL that dominates CAL Y also? Yes. How far can you go? Until the CAL is *tangent* to the investment opportunity set.
- Why cannot you pivot more than the tangent line? Because, at that point, you will have bypassed the entire investment opportunity set (minimum variance frontier) of risky assets. After all, we need to pair the risk-free asset with some *feasible* portfolio! Beyond the tangent, there are no more feasible portfolios to pair with the risk-free asset.
- The particular portfolio of X and Y at the point of tangency is called the *tangency portfolio*. In combination with the risk-free asset, it provides the CAL with the highest slope i.e. it provides the *maximum reward-to-risk ratio, or Sharpe ratio*. It is also sometimes called the *Mean Variance Efficient (MVE)* portfolio. It is the *optimal risky portfolio*.



The math of the MVE:

We have to solve an optimization problem here, where we find the portfolio that maximizes the Sharpe ratio or risk-to-reward ratio. This problem can be stated as:

$$\max_{w} \frac{E(\tilde{r}_p) - r_f}{\sigma_p}$$

$$subject\ to: E(\tilde{r}_p) = w.E(\tilde{r}_X) + (1-w).E(\tilde{r}_Y)$$

$$\sigma_p = \left[w_X^2 .\ \sigma^2(\tilde{r}_X) + w_Y^2 .\ \sigma^2(\tilde{r}_Y) + 2.w_X.w_Y.\rho_{X,Y}\sigma(\tilde{r}_X)\sigma(\tilde{r}_Y) \right]^{1/2}$$

The solution to this is the following messy expression:

$$w_X = \frac{\left[E(\tilde{r}_X) - r_f\right]\sigma_Y^2 - \left[E(\tilde{r}_Y) - r_f\right]Cov(\tilde{r}_X, \tilde{r}_Y)}{\left[E(\tilde{r}_X) - r_f\right]\sigma_Y^2 + \left[E(\tilde{r}_Y) - r_f\right]\sigma_X^2 - \left[E(\tilde{r}_X) - r_f + E(\tilde{r}_Y) - r_f\right]Cov(\tilde{r}_X, \tilde{r}_Y)}$$

$$w_Y = 1 - w_X$$

Plugging in our numbers into this equation results in w_X = 36.07% and w_Y = 63.93%. This implies a $E(\tilde{r}_{MVE}) = 16.39\%$ and a standard deviation, $\sigma(\tilde{r}_{MVE}) = 7.10\%$.

Look at the following table of Sharpe Ratios for our example:

Asset	$E(\tilde{r})$	$\sigma(\tilde{r})$	**Sharpe Ratio**
Asset X	0.10	0.07	(0.10-0.05)/0.07= 0.714
Asset Y	0.20	0.10	(0.20-0.05)/0.10 =1.500
MVE	0.1639	0.0710	(0.1639-0.05)/0.0710=**1.6036**

As we saw earlier from the graph, the Sharpe Ratio of the MVE is higher than either asset alone.

- **Three Risky Assets: An Illustrative example**
  - It is instructive to look at an example with 3 risky assets. The intuition from this example can be easily generalized to *N* risky assets.
  - Let us add one more risky asset to our risky assets (X and Y), and let's call it Z. We have to specify the expected return and variance of Z. Also, we need to specify the correlation (or equivalently the covariance) of Z with both X and Y.
  - It is useful to organize this information into *vectors* and *matrices*, as it can get out of hand pretty quickly as the number of assets increases. Matrix algebra also provides an elegant way to solve such portfolio optimization problems.
  - We write the mean return vector as: $\mu = \begin{bmatrix} E(\tilde{r}_X) \\ E(\tilde{r}_Y) \\ E(\tilde{r}_Z) \end{bmatrix} = \begin{bmatrix} 0.10 \\ 0.20 \\ 0.15 \end{bmatrix}$, and the matrix of variances and covariances as:

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_{X,Y} & \sigma_{X,Z} \\ \sigma_{X,Y} & \sigma_Y^2 & \sigma_{Y,Z} \\ \sigma_{X,Z} & \sigma_{Y,Z} & \sigma_Z^2 \end{pmatrix} = \begin{pmatrix} 0.0049 & 0.0007 & 0.0000 \\ 0.0007 & 0.0100 & 0.0108 \\ 0.0000 & 0.0108 & 0.0144 \end{pmatrix}$$

Notice that I have added another asset Z with an expected return of 15%, a standard deviation of 12%, having a correlation of 0 with Asset X, and a correlation of 0.9 with Asset Y.

- When you have three assets X, Y and Z, the investment opportunity set becomes all portfolios that can be formed from the three assets, i.e., an area rather than a line in mean-standard deviation space.
- Out of the infinite number of portfolios that we can form with the three assets, we have to find the portfolio that results in minimum possible risk, for each given level of expected return. Alternatively, there is *one portfolio* that results in the maximum expected return for each level of risk.

- We now have an optimization problem on our hands:

$$\min_{w_X, w_Y, w_Z} \sigma_p^2 = \left[w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + w_Z^2\sigma_Z^2 + 2w_X w_Y \sigma_{X,Y} + 2w_Y w_Z \sigma_{Y,Z} + 2w_X w_Z \sigma_{X,Z} \right]$$

$$\textit{subject to: } a)\ w_X.E(\tilde{r}_X) + w_Y.E(\tilde{r}_Y) + w_Z.E(\tilde{r}_Z) = m$$

$$b)\ w_X + w_Y + w_Z = 1$$

Here, we seek to minimize the objective function, the variance of the portfolio, which is the same thing as minimizing the standard deviation. Constraint a) says that this minimization is with respect to an expected return *m*. Constraint b) specifies that the weights have to add up to one.

- Using matrix algebra, we can solve this problem easily and elegantly to give us the minimum portfolio variance for each level of portfolio expected return *m*, as: $\hat{\sigma}_p^2 = \frac{1}{D}\left[C.m^2 - 2mB + A \right]$, where:

$$A = \mu'\Sigma^{-1}\mu$$

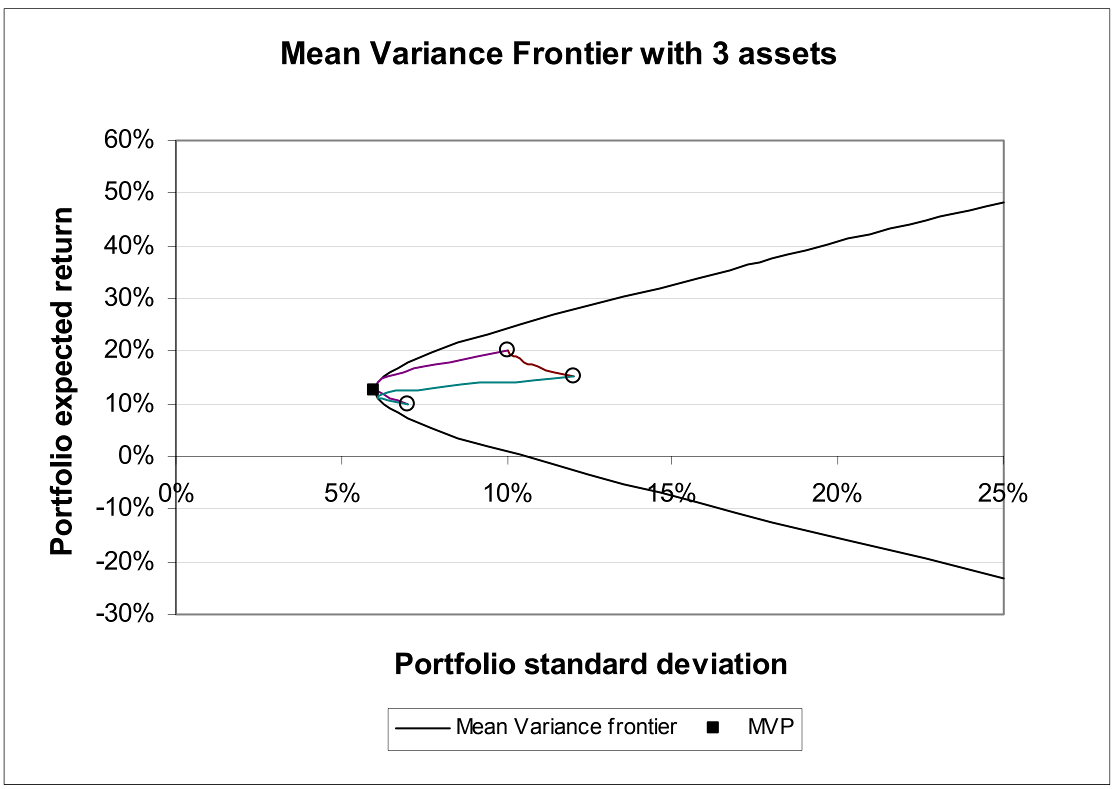
$$B = \mu'\Sigma^{-1}i = i'\Sigma^{-1}\mu$$

$$C = i'\Sigma^{-1}i$$

$$D = AC - B^2$$

Here, μ is the vector of asset expected returns, and i is a vector of ones. I do not expect you to know how to do this, but am providing these formulas only to give you a flavor of the results. (Come see me if you'd like to see the actual proof).

- The important thing is to *know* what the result above says. It says that for every level of portfolio return m, we can find the minimum possible variance using this formula. If we map out this equation in mean-standard deviation space, we see that it is a *parabola*.
- This parabola is called the *mean variance frontier*. Thus, the mean variance frontier plots all optimal combinations of risk and return in the presence of N risky assets. Let's look at the frontier for our three assets.



- In the above diagram, we can see the minimum variance frontier with respect to our three assets. This frontier is the envelope of all risk-return combinations of the three assets. It contains the mean variance frontiers formed by pair-wise combinations of the three assets.

- Now, on this MV frontier, the little black square is called the *global minimum variance portfolio* (a.k.a. MVP). This portfolio has the minimum variance of all possible portfolios formed from X,Y, and Z. The MVP has an expected return = $\frac{B}{C}$, and a standard deviation of $1/\sqrt{C}$. In our example, this works out to an expected portfolio return of 12.57% and a portfolio standard deviation of 5.98%.
- The part of the frontier that lies above the MVP is called the *efficient frontier*. For every portfolio on the efficient frontier, there is an inefficient portfolio with the same standard deviation and lower expected return directly below it.
- Thus, of the initial *feasible area*, we are left only with the northwest edge as the *efficient frontier*. In general, as we keep adding more and more assets, the efficient frontier will move in the northwesterly direction (why?)

> *A bit of history*[1]: It is this optimization problem that a young graduate student called Harry Markowitz set up and solved (not for 3 assets, but for *N* assets), at the University of Chicago in 1951. This was the dissertation he submitted for his Ph.D. degree. 39 years later, he would win the Nobel Prize for being the first to think in mathematical terms about risk and return. This seminal paper "Portfolio Selection" was published in the March 1952 issue of the *Journal of Finance.*
>
> In fact, his work was so radically mathematical for a paper in investments, that Milton Friedman, who was on his graduate committee (who would win the Nobel himself in 1976), said, "Harry, I don't see anything wrong with the math here, but I have a problem. This isn't a dissertation in economics, and we can't give you a Ph.D. in economics for a dissertation that's not economics. It's not mathematics, it's not economics, it's not even business administration." Needless to say, he did get his Ph.D.

[1] Extracted from *Capital Ideas – The improbable origins of Wall Street*, by Peter. L. Bernstein.

in economics. It is in honor of Markowitz that all the stuff we are studying is called "Markowitz Portfolio Theory".

- I will now state an important result (without proof) called *two-fund separation*. This result will be very useful very soon.
  - ✓ Two fund separation: All portfolios on the mean-variance efficient frontier can be formed as a weighted average of any two portfolios on the efficient frontier.
- Two fund separation has dramatic implications. According to this result, two mutual funds would be enough for all investors. There would be no need for investing in individual stocks separately; every investor could invest in a combination of these two funds (portfolios). But which two funds? Read on …

▪ **Diversification re-examined**

- So far, we have seen that by adding more and more assets, we can get more and more diversification and reduce portfolio variance and standard deviation. Can we ever eliminate all portfolio variance? In other words, can we reduce the portfolio variance to zero?
- Let us start from our general formula for portfolio variance:

$$\sigma^2(\tilde{r}_p) = \sum_{i=1}^{N} w_i^2.\sigma^2(\tilde{r}_i) + 2.\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} w_i.w_j.Cov(\tilde{r}_i.\tilde{r}_j)$$

- Since we are considering a portfolio of many assets, assume $w_i = \dfrac{1}{N}$, assume $\sigma^2(\tilde{r}_i) = \bar{\sigma}^2$, and $Cov(\tilde{r}_i, \tilde{r}_j) = \text{cov}$. Note that we have N variance terms and N^2-N covariance terms.
- This results in: $\sigma^2(\tilde{r}_p) = N.\dfrac{1}{N^2}\bar{\sigma}^2 + N(N-1)\dfrac{1}{N^2}\text{cov}$

$$= \frac{\bar{\sigma}^2}{N} + \frac{(N-1)}{N}\text{cov}$$

As $N \to \infty, \frac{1}{N} \to 0 \textit{ and } \frac{(N-1)}{N} \to 1$

$\Rightarrow \sigma_p^2 \to \text{cov}$

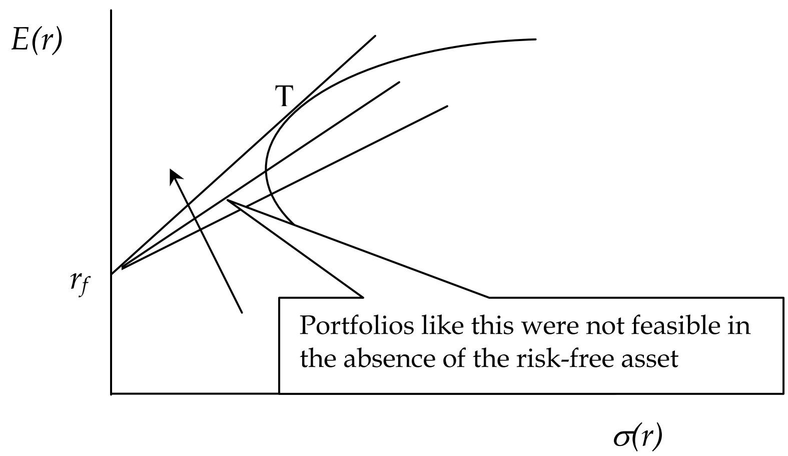
Conclusion 1: For a well-diversified portfolio, covariances matter, not so much the variances.

Conclusion 2: If the average covariance (equivalently, correlation) is not close to zero, we can never eliminate all risk. Even after diversifying a lot, we are still left with some residual risk.

The risk that we can get rid of just by diversifying is called *diversifiable* (or *unsystematic* or *idiosyncratic*) risk. e.g.: A fire at IBM's plant affects IBM stock, but this risk is unique to IBM stock. It can be gotten rid of by diversification.

The risk that remains even after diversifying is called *undiversifiable* (or *systematic* or *market-wide*) risk. e.g.: A change in interest rates or oil prices affects the whole economy. It cannot be gotten rid of by diversification.

- **Three Risky Assets and a risk-free asset**
  - Now that we have the efficient frontier with our three risky assets, we can throw our risk-free asset into the mix, and find the *Capital Allocation Line.*
  - The risk-free asset is noteworthy as it changes the shape of the efficient frontier from a *curve* (parabola) to a *straight line.*
  - There are many possible portfolios that can be formed by investing a proportion of the portfolio in the risk-free asset, and the remaining proportion in a risky portfolio that is on the efficient frontier.



Note the following from the diagram:

- We can form more portfolios than before because we now have a risk-free asset.
- Since we want to be as much to the northwest as possible, we will find it optimal to combine the risk-free asset with the highest possible point on the risky-asset efficient frontier.
- Geometrically, this means that we will draw a tangent from the risk-free asset to the efficient frontier, which is tangent at the *risky portfolio* T.
- *This tangent now becomes the new efficient frontier, because all points with optimal risk-return combinations (farthest to the northwest) lie along this tangent. This tangent is our **Capital Allocation Line (CAL)**.*
- How do we generate the CAL? Because of *two-fund separation*, we need any two assets (portfolios) on the CAL (which is the new efficient frontier). By forming portfolios of these two assets, we can generate the entire CAL. Which two assets or portfolios should we pick?
- Since the risk-free asset is by definition the zero-variance asset, it will be on the CAL. So we need one more (risky) portfolio to get the CAL. The natural and optimal risky portfolio to choose is the *tangency* or *MVE* portfolio.

- Once again, as for the case with two risky assets, we find the tangency, or MVE portfolio as the one which has the maximum *Sharpe Ratio* (slope of the CAL) among all the risky portfolios on the efficient frontier.
- This can be solved as a standard optimization problem, and the tangency portfolio is the one that has the following properties:

$$E(\tilde{r}_{MVE}) = \left(\frac{A - B.r_f}{B - C.r_f}\right), and\ \sigma(\tilde{r}_{MVE}) = \frac{\sqrt{H}}{(B - C.r_f)},$$
$$where\ H = C.r_f^2 - 2B.r_f + A$$

  For our example, I calculated $E(\tilde{r}_{MVE}) = 22.76\%$ and $\sigma(\tilde{r}_{MVE}) = 9.16\%$. This portfolio has $w_{X,MVE} = 22.74\%$, $w_{Y,MVE} = 177.93\%$, $w_{Z,MVE} = -100.67\%$.
- So much for the brute force way of getting the tangency portfolio. Is there a more intuitive way? Fortunately, there is. Remember that in trying to get the tangency portfolio, we are trying to find the portfolio with the maximum Sharpe Ratio.
- So if we start with an arbitrary portfolio of our three assets, by adding a little bit of (increasing the weight of) the asset with a high Sharpe ratio, and subtracting a little bit of (decreasing the weight of) an asset with a low Sharpe Ratio, we can improve the Sharpe Ratio of the overall portfolio. We can continue this process to its logical end – until the Sharpe ratios of portfolios of assets with the tangency portfolio are equal.
- We have seen that the only important determinant of portfolio variance (at the margin) is the covariance. So, we adjust portfolio weights until the ratio $\frac{E(\tilde{r}_i) - r_f}{Cov(\tilde{r}_i, \tilde{r}_{MVE})}$ is equal across all stocks. That is, in our 3-asset example, we are looking for the portfolio that has the property:

$$\frac{E(\tilde{r}_X) - r_f}{Cov(\tilde{r}_X, \tilde{r}_{MVE})} = \frac{E(\tilde{r}_Y) - r_f}{Cov(\tilde{r}_Y, \tilde{r}_{MVE})} = \frac{E(\tilde{r}_Z) - r_f}{Cov(\tilde{r}_Z, \tilde{r}_{MVE})}$$

- When a portfolio has this property, nothing can be gained by adding (or subtracting) an asset to (from) the portfolio. Such a portfolio must be the tangency portfolio. A proof of this assertion is provided next.

Proof: The ratio of risk premium to covariance with tangency portfolio is identical for every risky asset

Consider the situation when one is holding the MVE or tangency portfolio, T (with expected return $E(\tilde{r}_T)$ and variance σ_T^2. That means no other combination of risky assets can have a better reward to risk ratio (Sharpe ratio). We use this "maximum slope" property to derive the desired result.

Let's say we add a (very tiny) bit of GM stock to the MVE portfolio T. Specifically, consider a small dollar amount $\$\delta_{GM}$ *per each dollar* invested in T. Assume we finance this purchase by borrowing δ_{GM} at the risk-free rate r_f. Then the return of this changed portfolio C, is given by: $\tilde{r}_C = \tilde{r}_T + \delta_{GM}(\tilde{r}_{GM} - r_f)$.

The expected return and variance of portfolio C are given by:

$$E(\tilde{r}_C) = E(\tilde{r}_T) + \delta_{GM}[E(\tilde{r}_{GM}) - r_f] \quad \ldots \quad (1)$$

$$\sigma_C^2 = \sigma_T^2 + \delta_{GM}^2\sigma_{GM}^2 + 2\delta_{GM}Cov(\tilde{r}_{GM}, \tilde{r}_T) \quad \ldots \quad (2)$$

The change in expected return and variance due to the addition of GM is:

$$\Delta E(\tilde{r}) = E(\tilde{r}_C) - E(\tilde{r}_T) = \delta_{GM}[E(\tilde{r}_{GM}) - r_f] \quad \ldots \quad (3)$$

and

$$\begin{aligned}\Delta\sigma^2 = \sigma_C^2 - \sigma_T^2 &= \delta_{GM}^2\sigma_{GM}^2 + 2\delta_{GM}Cov(\tilde{r}_{GM}, \tilde{r}_T) \\ &\simeq 2\delta_{GM}Cov(\tilde{r}_{GM}, \tilde{r}_T)\end{aligned} \quad \ldots \quad (4)$$

where we note that since δ_{GM} is small, δ_{GM}^2 is smaller and close to zero.

Equation (4) says that no matter what the individual variance of GM, its contribution to overall portfolio variance is only through its *covariance with the* *tangency portfolio*. Thus, covariances, and not individual variances are relevant.

Now consider that, in a bid to improve the Sharpe ratio of the Tangency portfolio, we add a (very tiny) bit of GM stock, and subtract a tiny bit of IBM from the MVE portfolio T. Specifically, consider addition of a small dollar amount \$$\delta_{GM}$ *per each dollar* invested in T (as above) and subtraction of a small dollar amount \$$\delta_{IBM}$ *per each dollar* invested in T. We invest the proceeds of the IBM subtraction in the risk-free asset. Then the return of the changed portfolio C, is given by:

$$\tilde{r}_C = \tilde{r}_T + \delta_{GM}(\tilde{r}_{GM} - r_f) - \delta_{IBM}(\tilde{r}_{IBM} - r_f) .$$

Now, analogous to equations (1) and (2), we can write the expected return and variance of the changed portfolio C, as:

$$E(\tilde{r}_C) = E(\tilde{r}_T) + \delta_{GM}[E(\tilde{r}_{GM}) - r_f] - \delta_{IBM}[E(\tilde{r}_{IBM}) - r_f] \quad \ldots \quad (5)$$

$$\begin{aligned}\sigma_C^2 = \sigma_T^2 &+ \delta_{GM}^2\sigma_{GM}^2 + 2\delta_{GM}Cov(\tilde{r}_{GM}, \tilde{r}_T) \\ &+ \delta_{IBM}^2\sigma_{IBM}^2 - 2\delta_{IBM}Cov(\tilde{r}_{IBM}, \tilde{r}_T) \\ &- 2\delta_{GM}\delta_{IBM}Cov(\tilde{r}_{GM}, \tilde{r}_{IBM})\end{aligned}$$

which (using the same logic about δ_{GM} and δ_{IBM}) we can approximate as:

$$\sigma_C^2 = \sigma_T^2 + 2\delta_{GM}Cov(\tilde{r}_{GM}, \tilde{r}_T) - 2\delta_{IBM}Cov(\tilde{r}_{IBM}, \tilde{r}_T) \quad \ldots \quad (6)$$

As above, analogous to (3) and (4), we can write the changes in expected return and variance as:

$$\Delta E(\tilde{r}) = E(\tilde{r}_C) - E(\tilde{r}_T) = \delta_{GM}[E(\tilde{r}_{GM}) - r_f] - \delta_{IBM}[E(\tilde{r}_{IBM}) - r_f] \ldots \quad (7)$$

and $$\Delta\sigma^2 = \sigma_C^2 - \sigma_T^2 = 2\delta_{GM}Cov(\tilde{r}_{GM}, \tilde{r}_T) - 2\delta_{IBM}Cov(\tilde{r}_{IBM}, \tilde{r}_T) \quad \ldots \quad (8)$$

Now, in trying to improve the Sharpe ratio, let us try and keep adjust the δ_{GM} and δ_{IBM} such that there is no change in variance i.e. $\Delta\sigma^2 = \sigma_C^2 - \sigma_T^2 = 0$

Solving (8) for this yields: $$\delta_{IBM} = \frac{\delta_{GM}Cov(\tilde{r}_{GM}, \tilde{r}_T)}{Cov(\tilde{r}_{IBM}, \tilde{r}_T)} \quad \ldots \quad (9)$$

i.e. by adding exactly this amount of IBM, we are sure that our portfolio's variance does not increase because of the addition of GM and subtraction of IBM.

Let's see what the change in expected return is with these amounts of GM and IBM. IN other words, plug in the value of δ_{IBM} from (9) into (7). This yields;

$$\Delta E(\tilde{r}) = \delta_{GM}[E(\tilde{r}_{GM}) - r_f] - \frac{\delta_{GM} Cov(\tilde{r}_{GM}, \tilde{r}_T)}{Cov(\tilde{r}_{IBM}, \tilde{r}_T)}[E(\tilde{r}_{IBM}) - r_f],$$

which can be written as:

$$\Delta E(\tilde{r}) = \delta_{GM}\left[[E(\tilde{r}_{GM}) - r_f] - \frac{Cov(\tilde{r}_{GM}, \tilde{r}_T)}{Cov(\tilde{r}_{IBM}, \tilde{r}_T)}[E(\tilde{r}_{IBM}) - r_f]\right] \quad \ldots \quad (10)$$

Since we started at the Tangency (or MVE) portfolio T, it should be the case that the expression in equation (10) should be zero. Why?

If it is positive, it means that an improvement in expected return is possible without a concomitant increase in variance, which implies an improvement in the Sharpe Ratio, which in turn implies that the portfolio we began with is not MVE to begin with. *But we started with the MVE portfolio!*

Similar logic can be used to conclude that the expression in (10) cannot be less than zero. If it were, then one could add δ_{IBM} of IBM (given by (9)) and subtract δ_{GM} of GM to achieve a superior Sharpe Ratio, once again negating the premise that we started with the MVE portfolio. *But we started with the MVE portfolio!*

Therefore, it must be that (10) should evaluate to zero:

$$\Delta E(\tilde{r}) = \delta_{GM}\left[[E(\tilde{r}_{GM}) - r_f] - \frac{Cov(\tilde{r}_{GM}, \tilde{r}_T)}{Cov(\tilde{r}_{IBM}, \tilde{r}_T)}[E(\tilde{r}_{IBM}) - r_f]\right] = 0$$

$$\Rightarrow [E(\tilde{r}_{GM}) - r_f] = \frac{Cov(\tilde{r}_{GM}, \tilde{r}_T)}{Cov(\tilde{r}_{IBM}, \tilde{r}_T)}[E(\tilde{r}_{IBM}) - r_f]$$

$$\Rightarrow \frac{[E(\tilde{r}_{GM}) - r_f]}{Cov(\tilde{r}_{GM}, \tilde{r}_T)} = \frac{[E(\tilde{r}_{IBM}) - r_f]}{Cov(\tilde{r}_{IBM}, \tilde{r}_T)} \qquad \ldots \qquad (11)$$

We can use similar logic to conclude that this property (11) holds for all risky assets – not just GM and IBM.

Let us apply this property to our example, and obtain a more intuitive way of finding the tangency portfolio.

- <u>Procedure for finding the tangency portfolio</u>:

  <u>Step 1</u>: Form the following system of equations.

$$\begin{aligned} 0.0049w_X + 0.0007w_Y + 0.0000w_Z &= 0.10 - 0.05 \\ 0.0007w_X + 0.0100w_Y + 0.0108w_Z &= 0.20 - 0.05 \\ 0.0000w_X + 0.0108w_Y + 0.0144w_Z &= 0.15 - 0.05 \end{aligned}$$

  Solving this system yields the following:

$$w_X = 4.8186 \; w_{Y,MVE} = 37.6984, \; w_{Z,MVE} = -21.3294$$

  <u>Step 2</u>: Rescale the weights to add up to 1

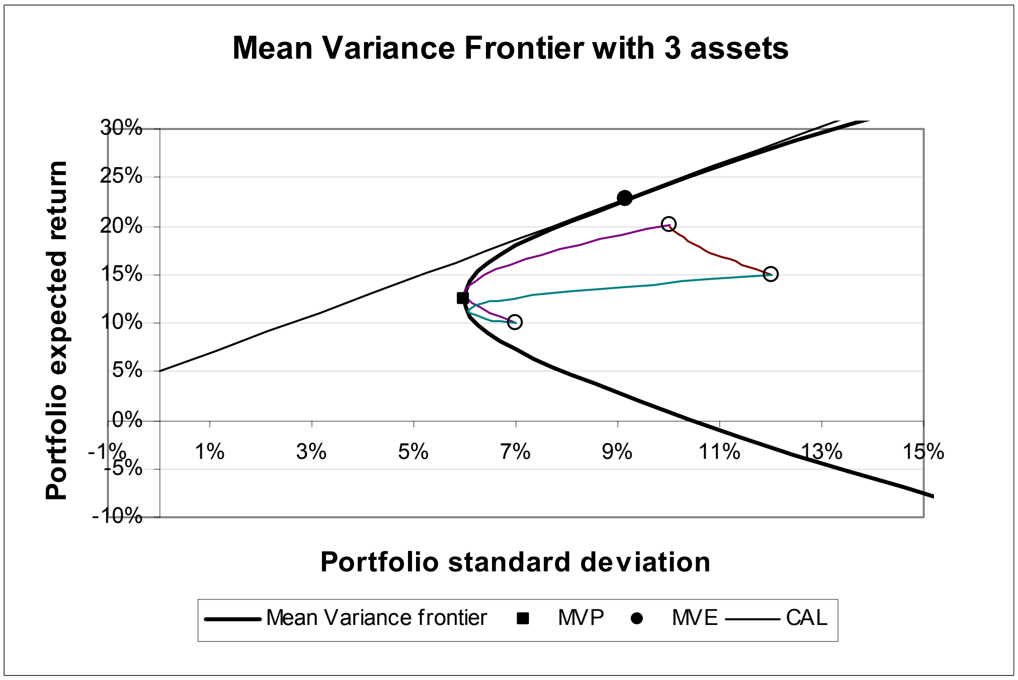
  $\Rightarrow w_{X,MVE} = 22.74\%, \; w_{Y,MVE} = 177.93\%, \; w_{Z,MVE} = -100.67\%$, the same values as we obtained before.

  Why is this method working?

  Think of a portfolio having $w_X, w_Y,$ and w_Z in each of the assets. The return on this portfolio is $w_X.r_X + w_Y.r_Y + w_Z.r_Z$. The covariance of Asset i with this portfolio is Cov(r_i, $w_X.r_X + w_Y.r_Y + w_Z.r_Z$) for i=X,Y,Z. So, the LHS of each of the above equations is the covariance of each stock with the portfolio. The RHS of each equation is the *risk premium* on that stock. By solving these equations, we are forcing the portfolio to have the property:

$$\frac{E(\tilde{r}_X)-r_f}{Cov(\tilde{r}_X,\tilde{r}_p)}=\frac{E(\tilde{r}_Y)-r_f}{Cov(\tilde{r}_Y,\tilde{r}_p)}=\frac{E(\tilde{r}_Z)-r_f}{Cov(\tilde{r}_Z,\tilde{r}_p)},$$ which as we saw just now, is the nice property of the MVE.

- Let's look at the CAL and the MVE in our example, with the help of a figure (I have zoomed in to the earlier figure and added the CAL):



- **Tying up loose ends: Integrating this chapter and the last**
  - In the previous lecture, we learned how to allocate capital between one risk-free asset and *one* risky asset. In this lecture, we learned how to come up with the *optimal risky portfolio*. Now, we shall put the two together in our example.
  - We retain the three risky assets from this lecture, along with the risk-free asset, and find the *optimal complete portfolio* (consisting of the risk-free asset, and the optimal risky portfolio) for an investor with quadratic utility and a coefficient of risk-aversion of A=40.

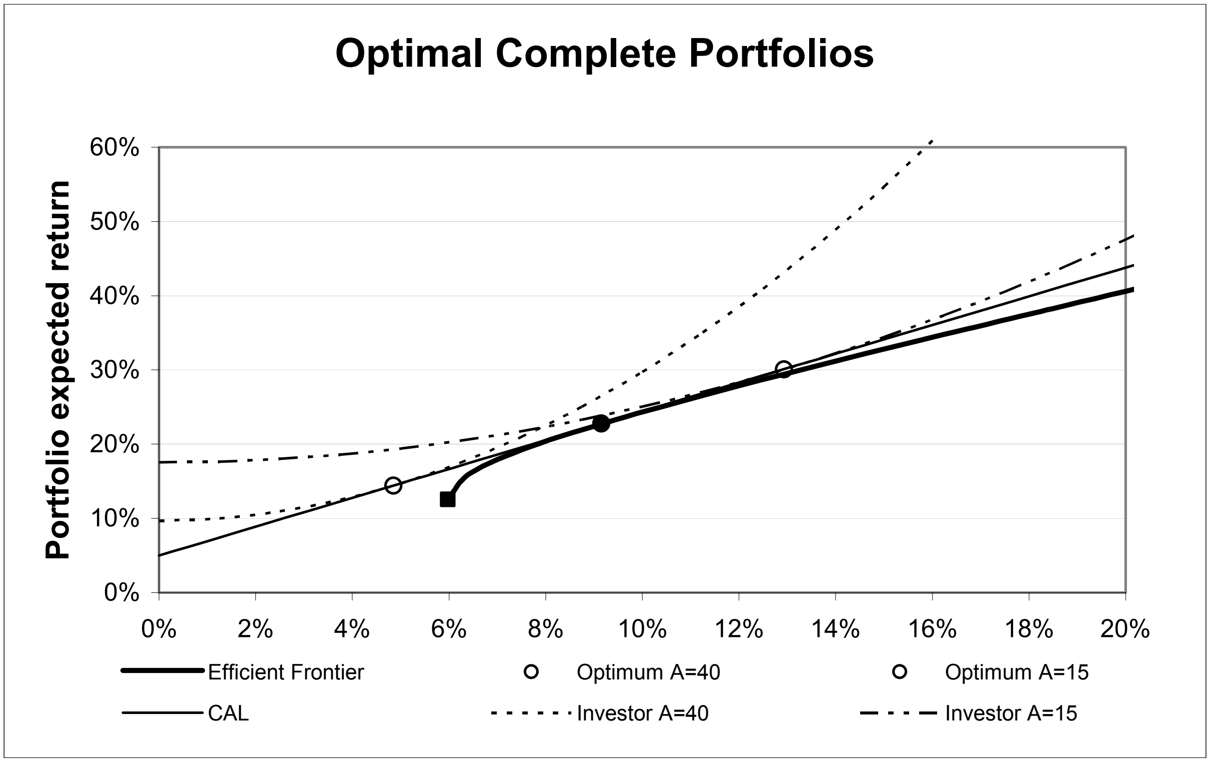
- Recall that the optimal position in the risky asset for a risk-averse investor was: $w^* = \frac{E(\tilde{r}) - r_f}{A.\sigma^2}$. This was equation (4) of the last lecture. Let us use this equation here. The $E(\tilde{r})$ and σ^2 of the above equation, correspond, of course, to the expected return and variance of the optimal risky portfolio. In our case, we know that this is the tangency portfolio, which means that: $w^* = \frac{0.2276 - 0.05}{40.(0.0916^2)} = 52.97\%$, and 1-$w^*$=1-0.5297=47.03%. This investor has an *optimal complete portfolio* with 52.97% invested in the optimal risky portfolio, and 47.03% in the risk-free asset.

- If the investor had an A=15, then the proportions in the risky and risk-free assets would be w^* = 141.25% and 1-w^*= -41.25% (Try this calculation!).

- Let's look at all these numbers in one place. The following table gives the allocation amounts and the optimal investments of both these investors in each security, assuming they each start with $1 million in capital.

Optimal complete portfolios		
Investment amount		**$1,000,000**
	Coeff. of risk aversion, A	
Allocation	40.0	15.0
Risk-free	0.4703	-0.4125
MVE	0.5297	1.4125
Final investments		
Risk-free	$470,309	-$412,509
X	$120,465	$321,240
Y	$942,460	$2,513,228
Z	-$533,234	-$1,421,958
Total	**$1,000,000**	**$1,000,000**

- Finally, let's look at both investors' indifference curves superimposed on the CAL.



- **Final Notes and Lead-in to next lecture**
  - Markowitz theory is elegant and correct if we are willing to assume that all investors care only about the mean and variance of asset returns.
  - This is true only if assume quadratic utility functions for all investors, or if returns are normally distributed. Both these assumptions are obviously very simplistic. Nevertheless, the basic lessons of Markowitz theory are valid: Portfolios are better than individual investments, and in large portfolios, covariances are all that matter. So, it remains the dominant paradigm of portfolio theory.
  - Two fund separation implies that an investment advisor should recommend the *same* risky portfolio to all investors regardless of their risk aversion. Depending on their risk aversion, investors adjust the proportions of their investment in the risk-free asset and the optimal risky

portfolio. This seemingly simple advice is quite radical, and flies in the face of much "practical investment advice".

- Again, problems crop up when applying the basic theory outlined in this lecture when our assumptions are invalid. i.e. when there are differential borrowing and lending rates, constraints on short sales etc. See sections 8.5 and 8.6 of BKM for illustrations on how this framework can be accommodated to incorporate such constraints.
- Finally, we have said nothing about where we get the expected returns, variances, and covariances required for Markowitz analysis. This is of course the task of the next part of the course. We shall try and understand models that "predict" expected returns: such models are called Asset Pricing Models.