FINC 748: Investments	**Ramana Sonti**
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Lecture note 3A: Capital Allocation between a risk-free and a risky asset

To be read with BKM Chapter 7

- One risky asset and one risk-free asset
- The capital allocation line
- Lending and borrowing portfolios
- Margin Transactions
- Optimal portfolio choice

- **One risk-free asset and one risky asset: Example from lecture 2**
  - Let us form a portfolio of the risk-free and risky assets from Lecture 2. Recall that the risky investment has an expected return, $E(\tilde{r})$ of 22%, and a variance $\sigma^2(\tilde{r})$ of 0.1176, which implies a standard deviation of $\sigma(\tilde{r}) = \sqrt{0.1176} = 34.29\%$. The risk-free asset has $E(\tilde{r}) = r_f$ =5%=0.05, and of course, a standard deviation of zero.
  - Let us combine these two assets in the proportion 60:40 i.e. let's invest 60% of our money in the risky asset and the other 40% in the risk-free asset. What about the expected return and standard deviation of this portfolio?
  - The following general formula is valid for a portfolio of a risk-free asset and any risky asset:
    - ✓ Portfolio return: $\tilde{r}_p = w_{risky}.\tilde{r}_{risky} + w_{risk-free}.r_f$

    Notes on Notation:

    1. Here $\tilde{r}_{risky}$ is the return of the risky asset, and r_f is of course the risk-free rate. For brevity's sake, I will drop the subscript, and use $\tilde{r}$ for the return on the risky asset.
    2. w_{risky} and $w_{risk-free}$ are the weights (or proportions or fractions) of the portfolio invested in the risky and risk-free assets respectively. Since the fractions of the total invested in the risky and risk-free assets have to add up to 1 or 100%, we have $w_{risk-free} = 1 - w_{risky}$. Henceforth in this lecture, I will drop the subscript and simply use *w* for the fraction of the portfolio invested in the risky asset, implying that the proportion of the total invested in the risk-free asset is (1-*w*).
    3. With these simplifications, the above equation reduces to:
       - ✓ Portfolio return $= \tilde{r}_p = w.\tilde{r} + (1-w).r_f$ (0)

4. Note that equation (0) indicates that the portfolio return $\tilde{r}_p$ is itself a *random variable*, which is a function of another random variable $\tilde{r}$.

- As for any random variable, we can calculate the expected value and variance (and hence, the standard deviation). Here, we can write the expected (mean) return and standard deviation of the portfolio as (more on how to derive these in Lecture 3B):
  - ✓ $E(\tilde{r}_p) = w.E(\tilde{r}) + (1-w).r_f$ (1)
  - ✓ $\sigma(\tilde{r}_p) = w.\sigma(\tilde{r})$ (2)

- Equation (1) can also be written as:
  - ✓ $E(\tilde{r}_p) = r_f + w(E(\tilde{r}) - r_f)$ (1A)

  This is not merely mathematical calisthenics. There is a neat interpretation that we can provide for equation (1A) as follows: The base rate of return for any portfolio is the risk-free rate, $\tilde{r}_f$. In addition, the portfolio is expected to return a *risk premium* that depends upon the risk premium of the risky asset, $E(\tilde{r}^e) = E(\tilde{r}) - r_f$, and the fraction of the portfolio invested in the risky asset.

- To return to our example, *w*=0.60, 1-*w*=0.40, $E(\tilde{r})$ = 22%, r_f =5%, $\sigma(\tilde{r}) = 34.29\%$. Now, we can substitute into the above equations:

  $E(\tilde{r}_p) = (0.6 \times 22\%) + (0.4 \times 5\%) = 15.2\%$, using equation (1)

  $\sigma(\tilde{r}_p) = 0.6 \times 34.29\% = 20.57\%$, using equation (2)

- Of course, we can also see that 15.2%=5%+0.6(22%-5%), using equation (1A), and the excess return interpretation.

- **Back to our investor**
  - What does this mean for our investor with a coefficient of risk aversion, *A*=3.0? Recall that given a choice between either the risk-free asset, or the risky asset, we concluded that this investor would be better off (have a higher utility) investing in the risk-free asset.
  - We can now calculate the utility she gets from our 60:40 portfolio of the risky and risk-free assets. Since the portfolio has an expected return $E(\tilde{r}_p) = 15.2\%$, and a standard deviation $\sigma(\tilde{r}_p) = 20.57\%$, we can calculate the $U_{portfolio} = 0.152 - \frac{1}{2}.3.(0.2057^2) = 8.85\%$. Obviously, our investor with *A*=3.0 would prefer the portfolio over either asset alone.
  - Clearly 60:40 is not the only *feasible* portfolio combination. There are many others (in fact an infinite number of combinations are possible). Any pair of real numbers (including negative numbers!) that add up to 1 is just fine to plug in as the proportions.
  - This suggests the following two-step decision making procedure for our investor.

    <u>Step 1</u>: Generate the *feasible set* of all possible combinations of the risk-free asset and the risky asset.

    <u>Step 2</u>: From all such feasible combinations, choose the one combination that gives her the highest utility. This particular combination is her *optimal portfolio*.
  - Let's work out both steps in detail.

- **The Capital Allocation Line**

- The Capital Allocation Line (CAL) plots all feasible portfolio combinations of the risky and risk-free assets.
- <u>Notation</u>: In what follows, I shall write $\sigma(\tilde{r}_p)$ as σ_p and $\sigma(\tilde{r})$ as σ.

- To derive the algebraic equation of the CAL, we need to algebraically relate the risk and return i.e. standard deviation and expected return of the portfolio. We can rewrite equation (2) as: $w = \frac{\sigma_p}{\sigma}$, and substitute this value of w into equation (1A) to yield:

  $$E(\tilde{r}_p) = r_f + w(E(\tilde{r}) - r_f) = r_f + (E(\tilde{r}) - r_f)\frac{\sigma_p}{\sigma}$$

  This is usually written as:

  $$E(\tilde{r}_p) = r_f + \left[\frac{E(\tilde{r}) - r_f}{\sigma}\right]\sigma_p \qquad (3)$$

  ✓ This is the equation of the *Capital Allocation Line*.

  For our example, the equation of the CAL reads:

  $$E(\tilde{r}_p) = 0.05 + \left[\frac{0.17}{0.3429}\right]\sigma_p$$

  From high school math, we know this equation is of the form:

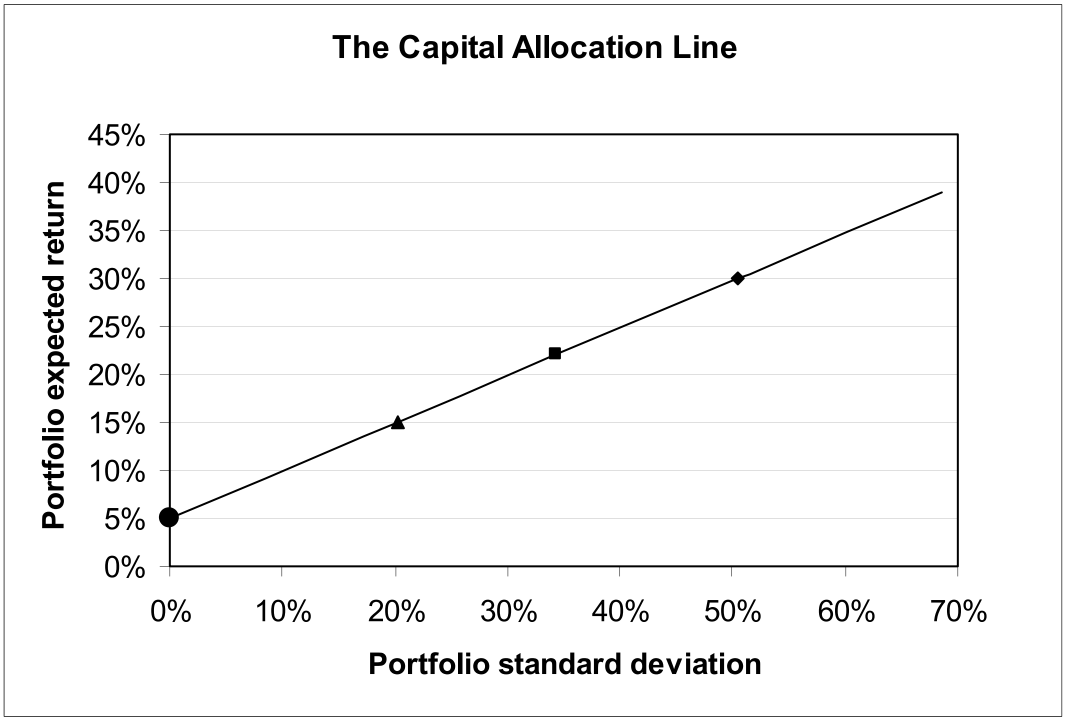
  y=mx+b,

  which is the equation of a straight line, with m as its slope and b the vertical intercept. This is why, mathematically speaking, the CAL is a straight line in the portfolio mean ($E(\tilde{r}_p)$)-standard deviation(σ_p)space

  <u>Graph of the CAL</u>

  To gain some better intuition, let us draw a graph of the CAL for our example. Since we know that the CAL is a straight line, we need only two points to draw it. Let us choose the two easiest points that we know *must* be on this line:

  1. A portfolio of 100% risky asset, 0% risk-free asset, in other words, the risky asset itself.
  2. A portfolio of 0% risky asset, 100% risk-free asset, in other words, the risk-free asset itself.



- The infinite number of points that make up this line correspond to the infinite number of feasible combinations of the risky and the risk-free assets in a portfolio.
- Among all these points, there are two points that interest us more than the others: the little black square in the above graph, which is our risky asset, and the little black circle, which is our risk-free asset. The latter plots on the vertical axis, as its standard deviation is zero.
- The slope of the CAL is $m=\left[\frac{E(\tilde{r})-r_f}{\sigma}\right]$. This quantity can be interpreted as the expected return per unit standard deviation. In other words, it is a reward-to-risk ratio. In finance, this term is usually called the *Sharpe Ratio*[1].

[1] In honor of Prof. William Sharpe of Stanford University, who was one of the researchers who came up with the Capital Asset Pricing Model (CAPM), a model that we shall study soon.

- Before we move on to Step 2, we should talk about the little triangle and the little diamond on the above figure, to get some more insight into the CAL.

- **Lending and borrowing portfolios**
  - Let's say (continuing with our risky and risk-free assets) investor Liza wants a portfolio with an expected return of 15%. How would she go about forming such a portfolio? Well, if she knows equation (1), she can solve for the proportions as follows:

$$E(\tilde{r}_p) = w.E(\tilde{r}) + (1-w).r_f = 15\%$$
$$\Rightarrow w(22\%) + (1-w)(5\%) = 15\%$$
$$\Rightarrow w = \frac{10}{17} = 58.82\%; 1-w = \frac{7}{17} = 41.18\%$$

How do we interpret these portfolio weights? This is straightforward. Liza invests 58.82% of her money in the risky asset, and the remaining 41.18% in the risk-free asset. The standard deviation of her portfolio can be found using equation (2): $\sigma_p = w.\sigma$ = 0.5882×34.29% = 20.17%.

The little black triangle represents Liza's portfolio in mean-standard deviation space. It is the point (20.17%, 15%).

  - Now, imagine another investor Barb who wants a portfolio with an expected return of 30%. How would she go about forming such a portfolio? Again, she uses equation (1) to solve for the proportions as follows:

$$E(\tilde{r}_p) = w.E(\tilde{r}) + (1-w).r_f = 30\%$$
$$\Rightarrow w(22\%) + (1-w)(5\%) = 30\%$$
$$\Rightarrow w = \frac{25}{17} = 147.06\%; 1-w = \frac{-8}{17} = -47.06\%$$

How do we interpret the negative portfolio weight?

- Remember a positive weight means investing in the risk-free asset, i.e. *lending money* to someone (most likely the U.S. Govt. by buying T-bills) and getting the risk-free rate of return. The converse of this statement says that a negative weight must mean *borrowing* money at the risk-free rate.
- This means that Barb has to do the following: Borrow $47.06 (at the risk-free rate of 5%), add this to her own money of $100, and invest the entire $147.06 in the risky asset. This enables her to shoot for a higher expected return (of 30%) than by investing all her money in the risky asset (which would have an expected return of 22%). This is called a *leveraged transaction*. Essentially, it means that one is investing OPM (other people's money), and it can be quite a risky affair. How risky? Let's see, using equation (2). The standard deviation of her portfolio is:

$$\sigma_p = w.\sigma = 1.4706 \times 34.29\% = 50.43\%.$$

  Obviously, Barb is exposed to much higher risk than Liza. The little black diamond represents Barb's portfolio in mean-standard deviation space. It is the point (50.43%, 30%).

- **Sidebar: Buying on Margin**
  - In practice, a leveraged transaction such as Barb's investment is quite straightforward. You can borrow up to a maximum 50% of your total investment from your broker, typically at close to the risk-free rate. So, if you put up $100 of your own money, you can invest upto $200, by borrowing $100 from your broker: i.e. $w = \frac{200}{100} = 2$, and 1-*w* = -1. This is called a *margin transaction*.
  - In this margin transaction, the amount of your own money relative to the total position is called the *percentage margin*. Current legal limit on *initial margin* is 50%, i.e. at most 50% of total position can be borrowed.

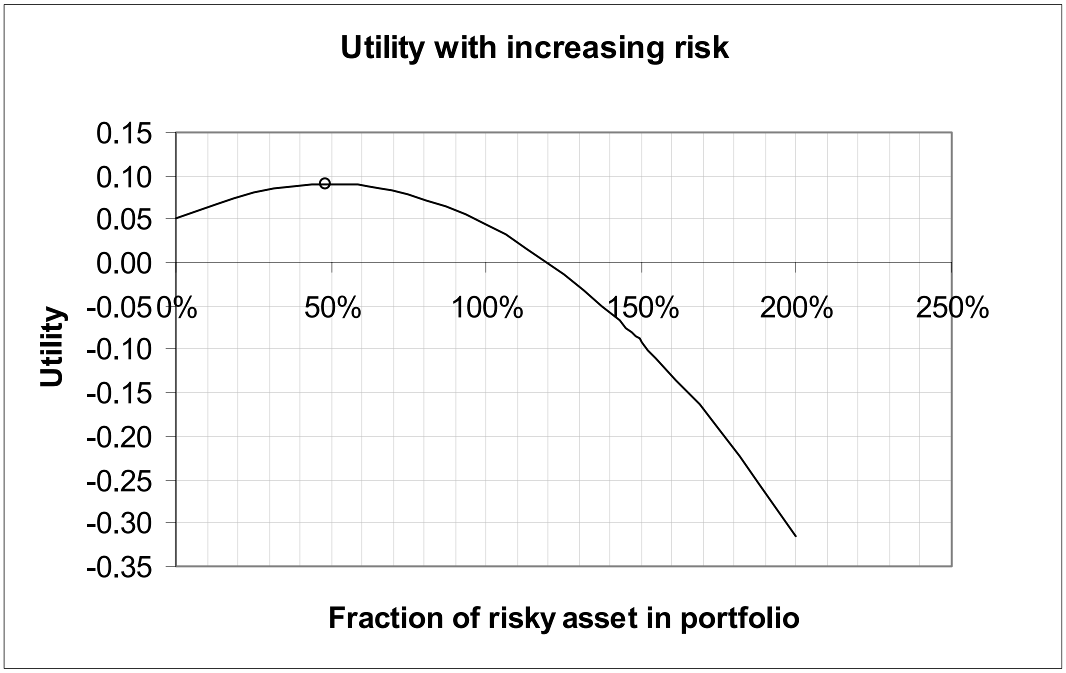
- There is also something called the *maintenance margin*, which is the percentage margin, or your equity in the total position to be maintained at all times. If you fall below the maintenance margin, the broker will issue a margin call and ask you to make up the difference in cash. See Section 3.6 on BKM, pp. 88-90 for a discussion on this issue.

▪ **Optimal portfolio choice**

- Here, we move into Step 2 of our decision making process. Having mapped out the feasible set of portfolios, we need to choose the one that maximizes our investor's utility. Let's do this graphically first, and then proceed to the algebra.
- Let's first form portfolios with varying weights on both our assets, and find the utility for each such portfolio (after applying equations (1) and (2) repeatedly for each portfolio). Calculations are shown below. Note that I have assumed a coefficient of risk aversion, *A*=3.0:

Utility from portfolio with *A* =			3.0	
Risky	**Risk-free**	***E*(*r*)**	**σ(*r*)**	**Utility**
0%	100%	5.00%	0.00%	0.0500
25%	75%	9.25%	8.57%	0.0815
48%	52%	13.19%	16.53%	0.0910
50%	50%	13.50%	17.15%	0.0909
59%	41%	15.00%	20.17%	0.0890
75%	25%	17.75%	25.72%	0.0783
100%	0%	22.00%	34.29%	0.0436
125%	-25%	26.25%	42.86%	-0.0131
147%	-47%	30.00%	50.43%	-0.0814
150%	-50%	30.50%	51.44%	-0.0918
175%	-75%	34.75%	60.01%	-0.1926
200%	-100%	39.00%	68.58%	-0.3155

- We can see that replacing the risk-free asset with the risky asset increases utility for a while … then, the additional expected returns from adding the risky asset are more than wiped out by the higher standard deviation associated with increasing risk. This point is better made by graphing the numbers from the above table.



- The maximum utility of U=0.0910 is reached when the fraction of the risky asset is w^*=48.19% (the little circle in the above figure). Thus, our investor with A=3.0 should invest 48.19% of her money in the risky asset, and the remaining 51.81% in the risk-free asset.
- What is the expected return and standard deviation of this *optimal portfolio*? Once again, we need to turn to equations (1) and (2)

  $E(\tilde{r}^*) = w^*.E(\tilde{r}) + (1 - w^*).r_f = (0.4819 \times 22\%) + (0.5181 \times 5\%) = 13.19\%$

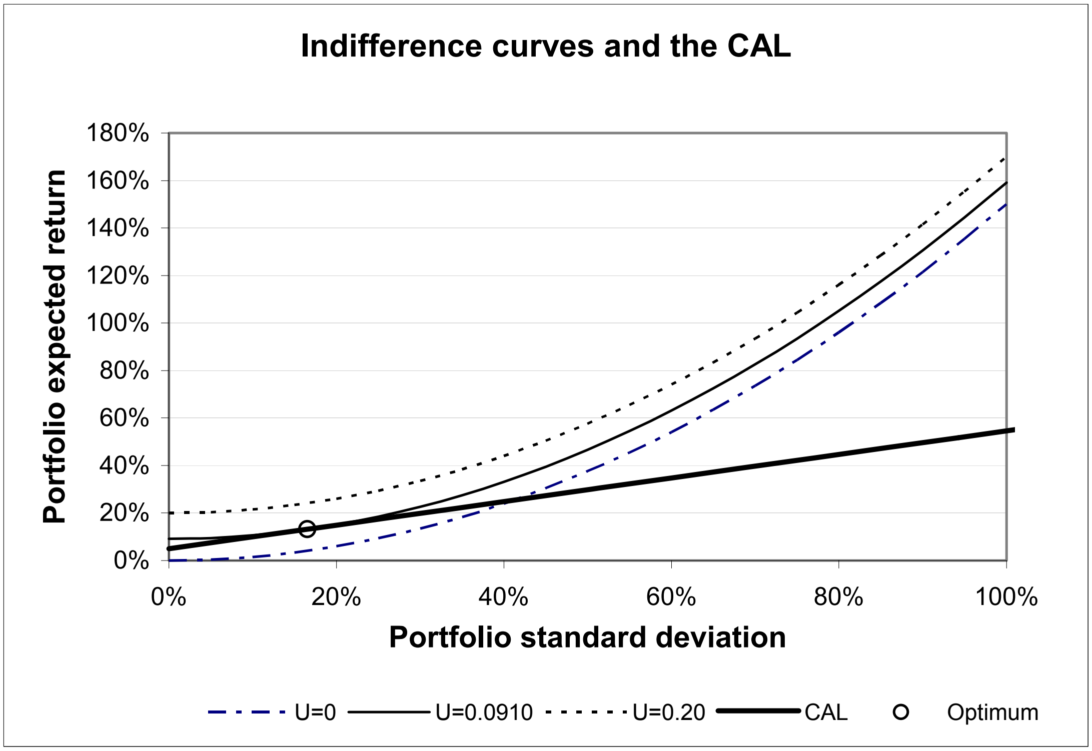
  and

  $\sigma_p^* = w^*.\sigma = (0.4819 \times 34.29\%) = 16.53\%$

  Just to double-check the result from the graph,

  $$\text{Utility, } U = 0.1319 - \frac{1}{2}(3)(0.1653^2) = 0.0910\text{, as advertised!}$$

- These very same points can be understood in elegant fashion if we illustrate the optimization process using indifference curves. Let us proceed to this next.



- In the above figure, I have reproduced the CAL from before. Now, I superimpose the indifference curves for our investor (with *A*=3.0) on this diagram. Recall that our investor wants to get as much to the northwest as she can.
- Start from *U*=0.0. At this point, the investor wants to get some utility. So she proceeds in the northwesterly direction stepping from one indifference curve to the next in search of more utility. (Remember indifference curves are parallel to each other). In time she reaches *U*=0.20, a utility of 0.20 units. However, she realizes that there are no portfolios in the feasible set that give her *U*=0.20. Ergo, *U*=0.20 is a pipe dream. Now she scales back her enthusiasm, and starts coming back.
- She backtracks until…you guessed it – until one of her indifference curves is *tangent* to the CAL. At that point she is at the maximum utility she can

get from this CAL. *The tangency portfolio is her optimal portfolio*. (The little circle in the above figure)

- **The math of the optimization**
  - The optimal portfolio is the solution to the following optimization problem:

$$\max_w U(\tilde{r}_p) = \max_w E(\tilde{r}_p) - \frac{1}{2}A\sigma_p^2$$
$$subject\ to: E(\tilde{r}_p) = r_f + w\left[E(\tilde{r}) - r_f\right]$$
$$\sigma_p^2 = w^2.\sigma^2$$

Substituting the *constraints* into the *objective function*, we can write the problem as: $\max_w r_f + w\left[E(\tilde{r}) - r_f\right] - \frac{1}{2}Aw^2.\sigma^2$

We proceed in the standard way of calculus: we find the first derivative of the objective function and set it equal to zero. The w that makes the derivative equal to zero is the optimal w, a.k.a. w^*.

i.e. w^* solves $\left[E(\tilde{r}) - r_f\right] - \frac{1}{2}A.2w.\sigma^2 = 0$

giving $w^* = \dfrac{E(\tilde{r}) - r_f}{A.\sigma^2}$ (4)

This solution[2] shows that the optimal position in the risky asset is:

✓ directly proportional to the risk premium on the risky asset

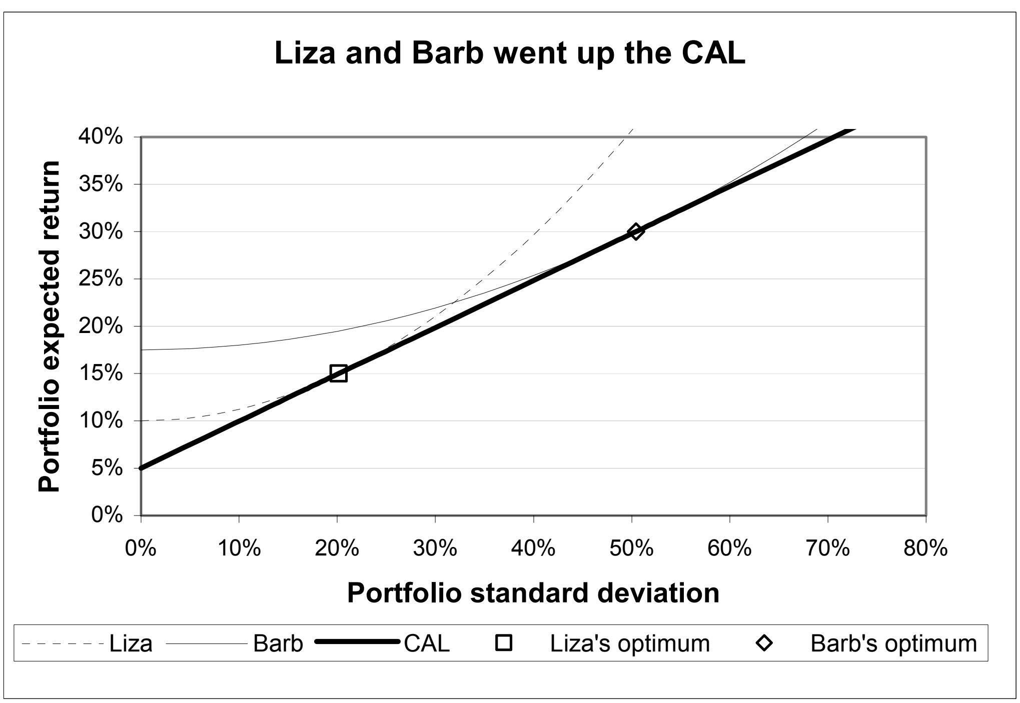
✓ inversely proportional to the level of risk of the risky asset

✓ inversely proportional to the level of risk aversion of the investor

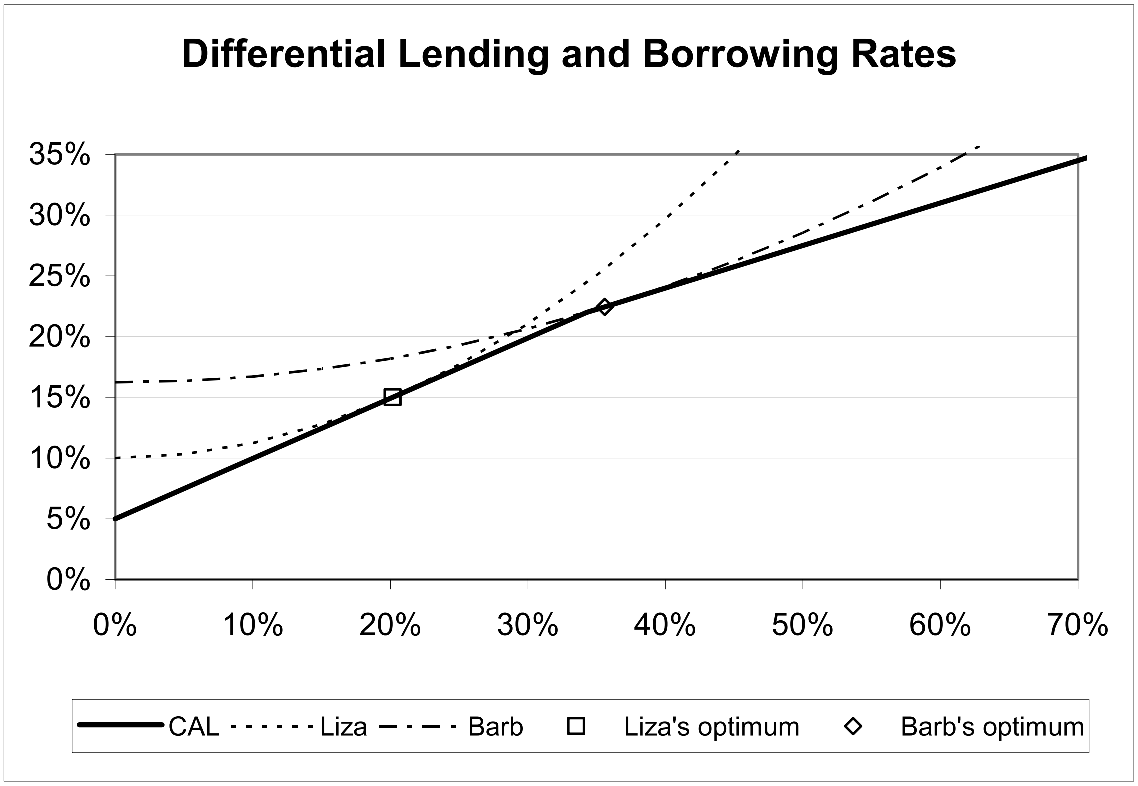
  - For our investor with A=3.0, we can use equation (4) to solve for the optimal portfolio: $w^* = \dfrac{E(\tilde{r}) - r_f}{A.\sigma^2} = \dfrac{0.22 - 0.05}{3.(0.3429^2)} = 0.4819 = 48.19\%$, the same number as we saw before.

[2] Again, as in Lecture 1, this formula differs from that in BKM because of the difference in convention of my entering fractions instead of percentage values directly. See endnote 2 of Lecture 1 notes for details.

- **To each her own optimum**
  - Remember that the optimal solution involves investor preferences and attitudes towards risk through the parameter *A*.
  - Let us consider our other two investors, Liza and Barb. Let's say Liza has a coefficient of risk aversion *A*=2.4579. Her optimal portfolio will have $w^* = \dfrac{0.22 - 0.05}{2.4579.(0.3429^2)} = 58.82\%$ in the risky asset. As we saw before this portfolio has an expected return of 15% and a standard deviation of 20.17%.
  - Consider Barb with a coefficient of risk aversion *A*=0.9832. This means that she is much less risk averse than Barb, and so her optimal portfolio will be more than a tad riskier. In fact, for Barb, $w^* = \dfrac{0.22 - 0.05}{0.9832.(0.3429^2)} = 147.06\%$ in the risky asset. As we saw before, this portfolio has an expected return of 30% and a standard deviation of 50.43%. As a final sanity check let us look at Liza and Barb's situation on the same CAL graph.



- **Endnote: Differential Lending and Borrowing Rates**
  - In the above analysis, we assumed that individuals can lend *and borrow* at the risk-free rate. Lending is fine, but what about borrowing? Can we really borrow at the risk-free rate?
  - When the borrowing rate is higher than the lending rate, the reward-to-risk ratio for those borrowing is lower. This leads to a kinked CAL. Let's see this in the context of Liza and Barb.
  - Let us keep the coefficients of risk aversion of Liza and Barb the same at A=2.4579, and 0.9832 respectively. We consider the same risk-free and risky assets, but add the caveat that the borrowing rate is r_b=10%, not r_f = 5% as earlier assumed.
  - This means that for all the investors who are lending (such as Liza), with $w < 1$, the slope of the CAL will still remain $\left[\frac{E(\tilde{r})-r_f}{\sigma}\right]$. However, for those borrowing (w>1) the slope will be $\left[\frac{E(\tilde{r})-r_b}{\sigma}\right]$, a lower reward-to-risk ratio.
  - This means that Liza's portfolio fraction in the risky asset, her expected return and standard deviation do not change. However, Barb's weight in the risky portfolio is now: $w^* = \frac{0.22-0.10}{0.9832.(0.3429^2)} = 103.81\%$, which is less than the 147.06% earlier (Why?). Her expected return is now 22.46% and the standard deviation of her portfolio return is now 35.60%. These numbers are lower than the earlier corresponding numbers of 30% and 50.43%.
  - Let's graph both these investors again in the context of the new (kinked) CAL.



- **Lead-in to next lecture**
  - In this lecture, we have learned how to allocate capital between one risk-free asset and *one* risky asset. How useful, and how general is this? The answer is: very!
  - Keep in mind that the one risky asset could be a portfolio, itself formed from many, many risky assets.
  - In fact, in the next lecture we shall deal exclusively with risky assets, and learn how to come up with the *optimal risky portfolio*. Then it will be time to put two and two together, and re-introduce the risk-free asset from this lecture.