# Pricing Options:
# The Binomial Way

FINC 456

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## Pricing Options:
## The important slide

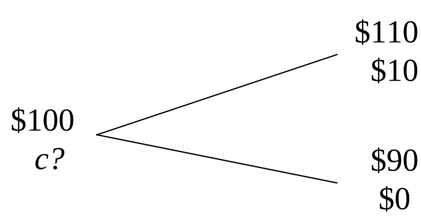
- Pricing options really boils down to three key concepts…
  - Two portfolios that have the same payoff cost the same.
    - Why?
  - A perfectly hedged portfolio earns the risk-free rate, because its payoff is the same regardless of the underlying asset's value
  - We can always form a perfectly hedged portfolio using the option and the underlying.
    - Why?

## Pricing Options: A simple example

- Consider a stock that is trading at $100 today, and will either be $110 or $90 in 3 months.
- What's the price of a call option with a strike price of $100, maturing in 3 months?
  - The risk-free rate is 8%
- The value of the option in 3 months is $10 or $0.
- Suppose I sell an option and buy .5 shares of stock…
  - If the stock goes up, I have $110(.5)-$10=$45
  - If the stock goes down, I have $90(.5)=$45
- The portfolio must be worth $\$45e^{-08(.25)}$=$44.11
- So the option value today is: (.5)$100 – c = $44.11
- The call price is $5.89 today…

## A closer look at the simple example: or, where'd that .5 come from…

- Usually, the example I just used is depicted graphically, by using a "tree"
  - Since there are only two outcomes, it is a *binomial tree*…




  - And I'm looking for the number of shares that I buy in order to "completely hedge" the call I sell
    - ✓ What do I mean completely hedge?

## Some notation…and that .5 answer

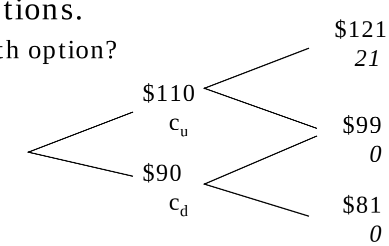
- Be clear on the notation:
  - S: the stock price today
  - u: the % move up in the stock
  - d: the % move down in the stock
  - $c_u$ and $c_d$: call option price if the stock is up or down
  - Δ: number of shares to buy to hedge portfolio
- So if we need the portfolio to be worth the same whether the stock goes up or down…

$$\Delta Su - c_u = \Delta Sd - c_d \Rightarrow \Delta = \frac{c_u - c_d}{Su - Sd}$$

- Or in our example, $\Delta 110 - 10 = \Delta 90 \Rightarrow \Delta = \frac{10}{20} = .5$

## No stopping us now…

- This principle can be easily extended to more and more complex situations.
  - How about a 6 month option?



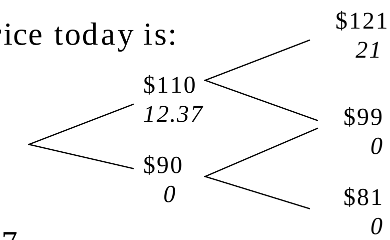

  - What is $c_u$? Well Δ is .9545, so

$$\Delta Su - c_u = e^{-.08(.25)} 94.5 \Rightarrow c_u = 12.37$$

  - How about $c_d$?

## And thus, for two periods…

- So then the option price today is:




- The $\Delta$ now is $\Delta = \frac{12.37}{20} = .6183$
- And thus $.6183(100) - c_u = e^{-.08(.25)} 55.648 \Rightarrow c_u = 7.28$
- There has to be an easier way…

## The binomial model: The elegant (and easy) way

- Let's go back to the equations…
  - We know the present value of the hedged portfolio
    - That's (pick the "up" state) $[Su\Delta - c_u]e^{-rT}$
    - How else can we interpret this?
  - We know the value of the portfolio today…
    - That's $\Delta S - c_u$
- So therefore $[Su\Delta - c_u]e^{-rT} = \Delta S - c_u$
- And since we know an equation for $\Delta$, we can substitute like mad…

$$c = e^{-rT}[qc_u + (1-q)c_d] \text{ where } q = \frac{e^{rT} - d}{u - d}$$

## The genius of risk-neutrality

- Consider the equation that we just derived…

$$c = e^{-rT}[qc_u + (1-q)c_d] \; where \; q = \frac{e^{rT} - d}{u - d}$$

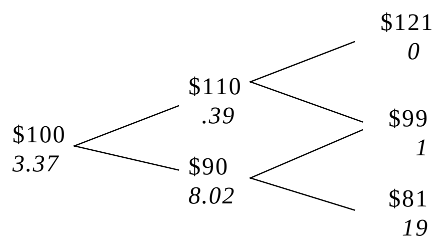
- First, let's see if it works… $q = .6010; e^{-rT} = .9802$
  - We still get $7.28 for the option, and it's lots faster…
- But what is q really?
  - Kind of looks like a probability…
  - In fact, it's the probability that makes the underlying grow at the risk-free rate.
    - ✓ We can assume this because we can always form a hedged portfolio of the derivative and its underlying security…

## Risk neutrality, con't

- What is the expected return on the stock with the "probability" $q$?
    - ✓ Note: what I'm calling $q$, Hull calls $p$
  - 
- In general, we can always use the assumption that individuals are risk-neutral and only demand the risk-free rate because they "could" hedge, and the risk-free rate is the best that they should earn…
  - Occasionally have to be careful, because what could destroy the ability to hedge?
    - ✓ Transactions costs, taxes, "lumpy" assets, etc…

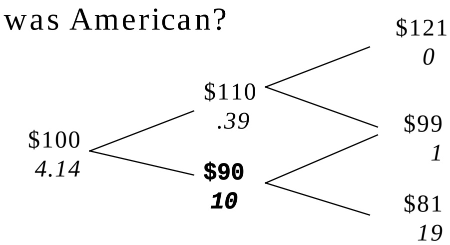
## Let's recap, with a put…

- Whether it's a put or a call, makes no difference
  - Really, all that changes is the payoff…
    - ✓ Which means we could price lot's of different stuff!
  - What about a put option struck at 100?
  - Well, q is still .6010 and the PV factor is .9802…




## How else could we have done it?

- Well, recall that
  $c - p = S - Xe^{-rT} \Rightarrow p = 7.28 - 100 + 96.09 = 3.37$
- But that's only for European options…what if the put was American?



# Back to the Delta

- Remember the delta of the option is the position in the underlying that created the riskless hedge…
  - This changes as the underlying price moves…
  - To hedge the option, one needs to trade continuously (and costlessly) in the underlying!
- Delta also has the interpretation as the change in the option price for a change in the underlying price.
- Hedging the risk of changes in the underlying price is called *rebalancing*, or delta hedging.
  - There needs to be borrowing and lending freely at the risk-free rate
  - If this is possible, we can replicate the payoff of the option by trading in the underlying and borrowing and lending

# Pricing Options: A dose of reality

- Where's the volatility? How accurate can this be?
  - The volatility is in the *u* and *d*…if the underlying can move a lot over a period, there's more volatility.
  - A given period is called a time step…
  - The idea is that greater accuracy is obtained by observing (and calculating) prices over smaller and smaller intervals
- If the time period is $h = \frac{T}{n}$ *where* $n =$ *number of steps*
- Then we can use q, u, and d to construct a tree…

$$u = e^{\sigma\sqrt{h}} \text{ and } d = e^{-\sigma\sqrt{h}}$$

$$q = \frac{e^{rh} - d}{u - d}$$

# Binomial Option Convergence

- Consider a call option with 6-month maturity, at the money (with the asset at $100), with volatility of .19 and a risk-free rate of 8%.