# The Binomial Option Pricing Model

This model, for the exact price of an option, is derived using the same principles that were used to derive boundary conditions and other option relationships such as the put-call parity. The major assumption is again that arbitrage opportunities should not prevail, so this type of option pricing is often called **arbitrage-free pricing.** But in order to derive an exact value for an option, we need to impose an assumption on the underlying stock price movement. We need not specify exactly where the stock price will be in the future, but different assumptions about possible stock price movements will result in different option valuation formulas.

The Binomial Option pricing formula allows the stock price to move to one of two possible prices over a single period. Although the single period model shown below does not seem realistic enough to be of any real use, it turns out that the multi-period binomial option pricing model is a very good approximation of the Black-Scholes model and, in fact, is exactly equal to the Black-Scholes model when the number of periods become infinite. Although the Black-Scholes model also restricts the possible movements of the underlying asset, it is the most widely used option pricing model in the financial world.

We derive several benefits from studying the binomial option pricing model. It helps us understand the basic principles on which option pricing is based. The mathematics for understanding the model is much less demanding than the level of mathematical sophistication required to fully understand the derivation of the Black-Ssholes model. But the principle behind the two models is exactly the same. Since the binomial model converges to the Black-Scholes model, we can use it directly to value options, and the answers are very close to what they would be, if you used the Black-Scholes formula. Often it is easier to modify the binomial model when you want to make it more realistic than it is to modify the Black-Scholes model. An example of this is the adjustments required for the model if the option is allowed to be exercised before expiration.

In this set of notes, we will describe one way to derive the model. It is important to understand the actual derivation, because that is how we can understand the principles behind the pricing of options, and learn how to benefit from mispriced options through the use of arbitrage. You should attempt to understand the actual derivations rather than memorizing formulas to plug into. The actual pricing needs to be done on a computer, so you are not being trained to crank out formula answers. It is important to learn when and how the model should be used.

## The Single Period Model

The single period model introduces the technique of the binomial option pricing model. The single period model is not meant to be a realistic technique to price real world options, but the results derived here can then be applied in the multi-period model. Assume we are one period from expiration. The length of the period does not matter for the purpose of demonstrating the model. The stock price is known one period before expiration. The strike price is also known. The stock price can have two possible returns between this period and the final period at expiration. The two possible returns must also be known, rate of interest is also known. Can we find the price of the call?

Let us introduce an example: Suppose that the stock price, one period before expiration is \$50, the strike price is \$50, and the stock can go up to \$100 (a return rate of 1.00 or 100%) or down to \$25 (a return rate of -0.50 or -50%). The risk free rate is 0.25 or 25%.

The stock price movement can be shown as:

| **one period before expiration** | **at expiration** | |
|---|---|---|
| | 100 | with probability q |
| 50 | | |
| | 25 | with probability 1-q |

We can calculate the two possible call values at expiration since we know the two possible stock prices at that time.

| | | |
|---|---|---|
| | Max[0,100-50] = 50 | with probability q |
| C | | |
| | Max[0, 25-50] = 0 | with probability 1-q |

Can we value this call? Consider a portfolio of 3 written calls, 2 shares of stock, and borrowing \$40. This borrowing is the same as issuing a pure discount bond with a price of \$40. With an interest rate of 0.25, the face value is \$50. Let us look at the cash flows for such a portfolio:

| Current Cash Flow | | Cash Flow at Expiration $S_T = 25$ | $S_T = 100$ |
|---|---|---|---|
| Write 3 Call | 3C | 0 | -150 |
| Buy 2 Shares | -100 | 50 | 200 |
| Sell a Bond | -40 | -50 | - 50 |
| | 3C-100+40 | 0 | 0 |

To avoid arbitrage situation,

$$3C-100+40 = 0$$

We can now solve for the call value which avoids arbitrage:

$$3C= 100-40$$
$$3C= 60$$
$$C=20$$

The call got the above problem must equal $20, or else there exits a riskless arbitrage opportunity. We will return to this problem later, but we now repeat the above without specific values for the variables, so that you can see how the proper portfolio can be obtained for any situation. The key knowledge is to know what proper proportions of stock and bonds are required per call that we are trying to value.

Let us introduce the following notation:

| | |
|---|---|
| $S$ | current stock price |
| $X$ | striking price |
| $S_u$ | stock price after one period if the stock price increases |
| $S_d$ | stock price after one period if the stock price decreases |
| $u$ | the rate of return on the stock if the stock price increases over one period |
| $d$ | the rate of return on the stock if the stock price decreases over one period |
| $r$ | the rate of interest on a riskless bond over one period |
| $C$ | current call value |
| $C_u$ | call value after one period if the stock price increases |
| $C_d$ | call value after one period if the stock price decreases |
| $q$ | the probability that the stock price will increase rather than decrease |
| $h$ | the number of shares of stock in a replicating portfolio of stocks and bonds which is designed to duplicate the cash flows to a call. |
| $B$ | dollar amount in riskless bonds in the above replicating portfolio |
| $n$ | end of the nth period, with n=0 being the point where we value the option. In the one period model, n=1 is at expiration, and n=0 is one period before expiration, and the point where we value the option. |

Consider the possible stock price movements over the last period before the call's expiration. The stock price at n=0 is S. It can either move up to price $S_u$ with probability q, or it can move down to price $S_d$ with probability 1-q. If it moves up, the rate of return on the stock will be u, and if it moves down, the rate will be d. the risk free rate of interest, r, must be between u and d. Why?

 

n=0 n=1

$$S_u = S(1+u) \quad \text{with probability q}$$

$$C$$

$$S_d = S(1+d) \quad \text{with probability 1-q}$$

We can calculate the two possible call values at expiration since we know the two possible stock prices at that time.

$$C_u = \max\left[0, S_u - X\right] \quad \text{with probability q}$$

$$C$$

$$C_d = \max\left[0, S_d - X\right] \quad \text{with probability 1-q}$$

The strategy for valuing the call is to consider a portfolio of h shares of stock and B dollars of riskless bonds. We know the value of such a portfolio after one period under our assumed stock price movements.

$$hS_u + (1+r)B \quad \text{with probability q}$$

$$hS_u + B$$

$$hS_u + (1+r)B \quad \text{with probability 1-q}$$

If we now select h and B such that:

$$hS_u + (1+r)B = C_u \qquad (1)$$

and

$$hS_d + (1+r)B = C_d \qquad (2)$$

To avoid riskless arbitrage opportunities:

$$C = hS + B \qquad (3)$$

The reason for this is the following: Since there is only one period left to expiration, if the two possible ending values of the call equal the two possible ending values of the portfolio of stocks and bonds, the value of the portfolio must also equal the value of the call. Otherwise, we have two ways of achieving the same cash flow, and we would always buy the cheaper one. Or we would buy the cheaper one, sell the more expensive one, and obtain a risk free arbitrage.

Notice that the two possible ending values of each portfolio are not the same. But if the stock price goes up, the call value and the replicating portfolio value are equal and the same thing if the stock goes down. Why are they equal? Because the particular values of h and B we have selected make them equal. So how do we find this particular set of values? We set the ending values of the call and the ending values of the replicating portfolio equal to each other, and h and B can be obtained from the simultaneous solution of equations (1) and (2):

First, solve for B from equation (2):

$$B = \frac{C_d - hS_d}{1 + r} \tag{4}$$

Put into equation (1):

$$C_u = hS_u + (1 + r)\left[\frac{C_d - hS_d}{1 + r}\right]$$

$$C_u = h(S_u - S_d) + C_d$$

$$h = \frac{C_u - C_d}{S_u - S_d} \tag{5}$$

Now, put equation (5) into equation (4), and simplify:

$$B = \frac{C_d - \left[\frac{C_u - C_d}{S_u - S_d}\right]S_d}{1 + r}$$

Multiply the $C_d$ term by $\frac{S_u - S_d}{S_u - S_d}$ which is equal to 1:

$$B = \frac{C_d\left[\frac{S_u - S_d}{S_u - S_d}\right] - \left[\frac{C_u - C_d}{S_u - S_d}\right]S_d}{1 + r}$$

$$B = \frac{C_d S_u - C_d S_d - C_u S_d - C_d S_d}{(S_u - S_d)(1 + r)}$$

$$B = \frac{C_d S_u - C_u S_d}{(S_u - S_d)(1 + r)} \qquad (6)$$

Now let us see if all these formulas work with our numerical example. Our numbers were:
S = 50
$S_u$ = 100
$S_d$ = 25
X= 50
r = .25
Since $S_u$ = S(1+u) and $S_d$ = S(1+d)
u= 1.0
d= -0.50
and since $C_u$ = Max (0, $S_u$ – X) and $C_d$= Max (0, $S_d$ – X)
$C_u$ = 50
$C_d$ = 0

Now calculating h and B:

$$h = \frac{C_u - C_d}{S_u - S_d} = \frac{50 - 0}{100 - 25} = 0.666667 \ \ shares \ / call$$
$$B = \frac{C_d S_u - C_u S_d}{(S_u - S_d)(1 + r)} = \frac{0(100) - 50(25)}{(100 - 25)(1.25)} = -13.333333 \ \ dollars \ / call$$

And using equation (3):

$$C = hS + B = (.666667)50 + (-13.33333) = 20$$

If we want to work in round shares of stock, we could just multiply the above by 1:
(.666667 shares/call) x (3/3) = 2 shares / 3 calls

(-13.3333333 dollars/call) x (3/3) = -40 dollars / 3 calls

We can now use our cash flow tables to demonstrate several interesting results. We have replicated a call with a portfolio of h stocks and B dollars in bonds. Since B is negative, it means that the bonds are used in an opposite manner from the stocks. That is,

 

if we buy the shares, we will need to sell bonds (this is the same as borrowing). If we sell the shares, we will buy bonds (same as lending money).
Writing 3 calls is the same as selling 2 shares and buying a bond for $40.

| Current Cash Flow | | Cash Flow at Expiration $S_T = 25$ | $S_T = 100$ |
|---|---|---|---|
| Write 3 Call | 60 | 0 | -150 |

| Current Cash Flow | | Cash Flow at Expiration $S_T = 25$ | $S_T = 100$ |
|---|---|---|---|
| Selling 2 shares | 100 | - 50 | -200 |
| Buy bond (lend 40) | - 40 | 50 | 50 |
| | 60 | 0 | -150 |

These cash flow diagrams show that the two portfolios are the same. What arbitrage would you do if C= 18 instead of 20?

The theory says that $C = hS + B$. If $C < hS + B$ we should buy a call and sell the portfolio of hS and B. But since B is negative, this means we buy bonds (lend) to get the required cash flow.

| Current Cash Flow | | Cash Flow at Expiration $S_T = 25$ | $S_T = 100$ |
|---|---|---|---|
| Buy 3 calls | 54 | 0 | 150 |
| Selling 2 shares | 100 | - 50 | -200 |
| Buy bond (lend 40) | - 40 | 50 | 50 |
| | 6 | 0 | 0 |

As an exercise, demonstrate that buying 3 calls is the same as buying two shares of stock and selling a bond for $40 (borrowing). Also show what you would do if the market price of the call is $28 instead of $20.

Notice that the variable 1 does not enter any of our calculations. We do not care what the probabilities are for an increase versus a decrease in the stock price. The reason for this is that we are pricing the call strictly as an arbitrage with respect to the stock price. Option pricing as we have shown it, is a relative valuation technique. q probably will influence the present price of the stock, which will affect the value of the call. But for a given stock price, q will not change the value of the call. Another way to see this is to be convinced that if we are holding a hedged portfolio of calls and stocks, we do not care if the stock price goes up or down.

 

**Homework**

Consider the following one period binomial model example. The stock price is at 100 and could either go up to 140 or down to 90. The exercise price is at 80. Let the risk free rate of return be 20% over this period.

a. Calculate u, d, r, $C_u$, $C_d$ and then calculate $B_0$ and h.

b. Calculate $C_0$ and demonstrate with an arbitrage table that this is the proper value if there is to be no arbitrage opportunities.

c. If the market price of this option is $30 what would you do? Again, use an arbitrage table and demonstrate that there is now an arbitrage profit available.

d. What would you do if the market price is $40?

 