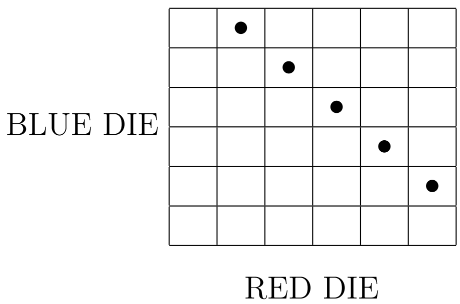
1 Agreeing to disagree

Aumann, in '81, qualifies correlated equilibria as expression of bayesian rationality.
To understand and discuss the content of Aumann's assertion, we need some further technical tools. In particular, we need a richer language.

I will begin with an example:
You are a D.M.
What you get depends on : $\begin{cases} a \in A & \leftarrow \quad \text{your action} \\ \omega \in \Omega & \leftarrow \quad \text{true state of nature} \end{cases}$

Typical problem of decision under incertanty.
You must choose a before knowing ω
Consider an example. You are offered to bet on the result of a throw of couple of dice (for what will follow, I assume that one is red and the other is blue).
You will gain G if the sum of dice is 8, and you will pay L otherwise.
Here a reasonable (not the unique which is possible or reasonable) representation of Ω is $\Omega = \{1, \dots, 6\}^2$.



With $p(i,j) = \frac{1}{36} \quad \forall (i,j) \in \Omega$
Assume that you are simply an expected money maximizer. Your vN-M utility function is (linear with) money.

TO BET: $\frac{5}{36}G - \frac{31}{36}L$
NOT TO BET: 0

So, you will bet if $\frac{5}{36}G \geq 31L\ldots$
Notice that your choice is *NOT CONDITIONED* upon ω. Obviously. You don't know ω. We are obeying to some minimal realism assumption.

Would be different if you knew ω. Of course, if $\omega \in \{(2,6),(3,5),\dots,(6,2)\}$ then you would "bet", getting G. Otherwise you would not bet, getting 0. Not serious.
But there are interesting "intermediate" cases.
For example, you could be allowed to see the result of ONE die just before betting. More precisely, let's assume that you can see the result of the red die.
This means that you have partial information. This fact can be represented by means of an information partition
$\mathcal{P} = \{\{(1,2),\dots,(1,6)\},\{(2,1),\dots,(2,6)\},\dots,\{(6,1),\dots,(6,6)\}\} = \{P_1,\dots,P_6\}$.
This (info partition) is a standard tool.
The interpretation is obvious. If ω is the true state of nature, the DM knows only $P(\omega)$, the element of the partition to which ω belongs.
So, the action of DM can be contingent on $P(\omega)$.
Of course, to decide, the DM will re-compute the probably distribution based on his partial information.
6 cases:
$\omega = (1,j)$, i.e. we are in P_1.
The probability that the sum is 8 is zero. so:
TO BET: $-L$
NOT TO BET: 0.
$\omega = (k,j)$; i.e. we are in $P_k, k = 2,\dots,6$, the probability that the sum is 8 is $\frac{1}{6}$. So:
TO BET: $\frac{1}{6}G - \frac{5}{6}L$
NOT TO BET: 0.
All of this with just one DM.
If the DMs are two (or more)? Clearly, the key issue here is that they may have different (partial) information.
For example, DM_2 could know the result of the "second" die, the blue one.
So, he has a different information partition. To avoid confusion, we shall call $\mathcal{P}_1$ the information partition of DM_1 and $\mathcal{P}_2$ the information partition of DM_2.
$\mathcal{P}_2 = \{\{(1,1),\dots,(6,1)\},\{(1,2),\dots,(6,2)\},\dots,\{(1,6),\dots,(6,6)\}\}$
For example, if the true ω is $(1,3)$, w.r.t. to the bet:
1 assigns $prob = 0$ to the event E that the sum of dice is 8.
2 assigns $prob = \frac{1}{6}$ to the same event E.
So, if we have that $\frac{1}{6}G - \frac{5}{6}L > 0, DM_2$ will bet, while 1 not.
Nothing strange...

Notice the following.

If 2 knows that 1 assigns $prob = 0$ to the event E, than 2 will further revise his probability assessments!
NOTICE that for this to happen, it is essential that player 2 *KNOWS* $\mathcal{P}_2$, the info partition of 1 (or that, at least, has some info about that).
So, the fact that 1 and 2 have different beliefs about E, cannot be a shared, a common information.
This is the key point of Aumann's "agreeing to disagree".

Please, notice that this was just a simple example. In particular, it was enough for player 2 to know the probability assigned by 1. One can construct more sophisticated examples, with more elaborate knowledge interactions.
I will turn now to a very sketchy introduction to the formalism of CK, just to have the minimal instruments for understanding both "agreeing to disagree" and "correlated equilibria as expression of bayesian rationality".

Formalism and preliminaries

Here are some preliminaries and a theorem that we need in the proof of Aumann's theorem.

I will follow Osborne and Rubinstein, Chapter 5.
We have Ω (finite, always, to simplify techniques) and $\mathcal{P}_1, \mathcal{P}_2$ two (information) partitions of Ω.
Notice that a partition $\mathcal{P}_i$ $(i = 1, 2)$ identifies an information function P_i, in a obvious way:

$P_i : \Omega \to 2^\Omega \setminus \{\emptyset\}$. ($2^\Omega$ denotes the set of all subsets of Ω)

$P_i(\omega)$ is just the set of $\mathcal{P}_i$ who contains ω.
We shall say that an event $F \subseteq \Omega$ is *SELF EVIDENT* between 1 and 2 if *FOR ALL* $\omega \in F$ we have that $P_i(\omega) \subseteq F, i = 1, 2$.
An event $E \subseteq \Omega$ is CK between 1 and 2 *IN THE STATE* $\omega \in \Omega$ if there is a self-evident event F st: $\omega \in F \subseteq E$.
Notice that the following result holds.

Theorem 1 *Given $\Omega, \mathcal{P}_1, \mathcal{P}_2$ and an event F, the following are equivalent:*
1) F is self-evident between 1 and 2.
2) F is union of members of the partitions $\mathcal{P}_i, i = 1, 2$

Proof.
1) $\Rightarrow$ 2)
Because $\forall \omega \in F, \mathcal{P}_i(\omega) \subseteq F$ for $i = 1, 2$ we have that $E = \cup_{\omega \in E} P_i(\omega)$, for $i = 1, 2$.
Notice that $P_i(\omega)$ is an element of the partition $\mathcal{P}_i$, due to the way in which we defined P_i.

2) $\Rightarrow$ 1)
Since
$F = \cup_{\alpha \in A} P_{1,\alpha}$ with $P_{1,\alpha} \in \mathcal{P}_1 \quad \forall \alpha \in A$ and
$F = \cup_{\beta \in B} P_{2,\beta}$ with $P_{2,\beta} \in \mathcal{P}_2 \quad \forall \beta \in B$,
clearly, every $\omega \in F$ will be in some $P_{1,\alpha}$ (with $P_{1,\alpha} \subseteq F$) and in some $P_{2,\alpha}$ (again, with $P_{2,\alpha} \subseteq F$).
So, F is self-evident (between 1 and 2).
Notice that I have skipped completely the interesting issue of links between knowledge operation and information functions. See, once more, chapter 5 of O-R.

Aumann's theorem and its proof

We have Ω (finite), and p, a probability distribution on Ω (to be interpreted later as the "common prior").
We remind that a function $P : \Omega \to 2^{\Omega} \setminus \{\emptyset\}$ is said to be an *information function*. We shall assume that P satisfies the following conditions:

$\omega \in P(\omega)$ for all $\omega \in \Omega$
if $\omega' \in P(\omega)$, then $P(\omega) = P(\omega')$

It can be shown that P is "partitional" (i.e., there is a partition such that for all $\omega \in \Omega$, $P(\omega)$ is just the element of the partition containing ω) if and only if P satisfies the two conditions above.

Let P be an information function and let E be an event.
Given $\omega \in \Omega$, at ω the DM will assign to E the probability

$$p(E|P(\omega))$$

(i.e. the probability of E, conditional on $P(\omega)$).
In an example, E was the event: sum of dice =8.
And, for example, at $\omega = (1, 3)$ we had $p(E|P_1(\omega)) = 0 \quad p(E|P_2(\omega)) = \frac{1}{6}$.

Remark: the event that "DM i assigns the probability p_i to E" is:

$$\{\omega \in \Omega : p(E|P_i(\omega)) = p_i\}.$$

Theorem 2 *It is given Ω finite and a probability p on Ω (the common prior). We are given two information functions P_1 and P_2.*
Assume that it is CK between 1 and 2 in some state $\omega^\star \in \Omega$ that 1 assigns probability p_1 to same event E and that 2 assigns probability p_2 to E.
Then, $p_1 = p_2$

Proof. The event "1 assigns probability p_1 to E and 2 assigns probability p_2 to E " is:

$$\{\omega \in \Omega : p(E|P_1(\omega)) = p_1\} \cap \{\omega \in \Omega : p(E|P_2(\omega)) = p_2\}$$

Since it is assumed to be CK, there is a self evident F s.t.:

$$\omega^\star \in F \subseteq \{\underbrace{\omega \in \Omega : p(E|P_1(\omega)) = p_1}_{*}\} \cap \{\omega \in \Omega : p(E|P_2(\omega)) = p_2\}$$

Thanks to the theorem proved above, we have that F is a union of members of the partition P_1 and P_2.
So, $F = \cup_{\alpha \in A} P_{1,\alpha} = \cup_{\beta \in B} P_{2,\beta}$.
Now, notice that, for every $\alpha \in A$, $p(E|P_{1,\alpha}) = p_1$: to be sure of that, it is enough to notice that $P_{1,\alpha}$ is one of the $P_1(\omega)$ that appear in $*$.
In more detail:
Take $\omega \in F$.
Because $\omega \in *$, we have that $p(E|P_1(\omega)) = p_1$.
But $\omega \in F$, so $P(\omega)$ is one of the elements of the info partition whose union gives F. That is, $P_1(\omega) = P_{1,\alpha}$ for some $\alpha \in A$.

So, $p(E|P_{1,\alpha}) = p_1$ for every $\alpha \in A$.
Hence, $p(E| \cup_{\alpha \in A} P_{1,\alpha}) = p_1$

Namely, [$p(E|P_{1,\alpha'} = p_1$ and $p(E|P_{1,\alpha''}) = p_1$] IMPLIES that $p(E|P_{1,\alpha'} \cup P_{1,\alpha''}) = p_1$

So, $p(E|F) = p$.
But the same reasoning can be repeated for p_2. So, we get $p(E|F) = p_2$.
But $p(E|F)$ is a well defined number ...